## 0.4 Exercises

**0.1** Find all functions $x(t), y(t)$ satisfying

$$x'(t) = y(t) - x(t),$$

$$y'(t) = 3x(t) - 3y(t).$$

Find the particular pair of functions satisfying $x(0) = y(0) = 1/2$.

**0.3** The Fibonacci numbers $F_n$ are defined by $F_1 = 1$, $F_2 = 1$ and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$. Find a formula for $F_n$ by solving the difference equation.

**0.4** Find the function $f(n)$, $n = 0, 1, 2, \ldots$ that satisfies

$$f(0) = 0,$$

$$f(n) = \frac{1}{3}f(n-1) + \frac{1}{3}f(n+1) + \frac{1}{3}f(n+2), \quad n \geq 1,$$

$$\lim_{n\to\infty} f(n) = 1.$$

**0.5** Find all functions $f$ from the integers to the real numbers satisfying

$$f(n) = \frac{1}{2}f(n+1) + \frac{1}{2}f(n-1) - 1. \tag{0.8}$$

[Hint: First show that $f(n) = n^2$ satisfies (0.8). Then suppose $f_1(n)$ and $f_2(n)$ both satisfy (0.8) and find the equation that $g(n) = f_2(n) - f_1(n)$ satisfies.]

**0.6** (a) Find all functions $f$ from the real numbers to the real numbers such that for all $x$,

$$f''(x) + f'(x) + f(x) = 0.$$

(b) Find all functions $f$ from the integers to the real numbers such that for all $n$,

$$f(n+2) = -f(n) - f(n+1).$$

**1.1** The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the papers in the pile and puts them in the recycling bin. Also, if ever there are at least five papers in the pile, Mr. Smith (with probability 1) takes the papers to the bin. Consider the number of papers in the pile in the evening. Is it reasonable to model this by a Markov chain? If so, what are the state space and transition matrix?

**1.2** Consider a Markov chain with state space $\{0, 1\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix} \end{array}.$$

Assuming that the chain starts in state 0 at time $n = 0$, what is the probability that it is in state 1 at time $n = 3$?

**1.3** Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .4 & .2 & .4 \\ .6 & 0 & .4 \\ .2 & .5 & .3 \end{bmatrix} \end{array}.$$

What is the probability in the long run that the chain is in state 1? Solve this problem two different ways: 1) by raising the matrix to a high power; and 2) by directly computing the invariant probability vector as a left eigenvector.

**1.5** Consider the Markov chain with state space $S = \{0, \dots, 5\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[ \begin{array}{cccccc} .5 & .5 & 0 & 0 & 0 & 0 \\ .3 & .7 & 0 & 0 & 0 & 0 \\ 0 & 0 & .1 & 0 & .9 & 0 \\ .25 & .25 & 0 & 0 & .25 & .25 \\ 0 & 0 & .7 & 0 & .3 & 0 \\ 0 & .2 & 0 & .2 & .2 & .4 \end{array} \right] \end{array}.$$

What are the communication classes? Which ones are recurrent and which are transient? Suppose the system starts in state 0. What is the probability that it will be in state 0 at some large time? Answer the same question assuming the system starts in state 5.

**1.7** Let $X_n$ be an irreducible Markov chain on the state space $\{1, \dots, N\}$. Show that there exist $C < \infty$ and $\rho < 1$ such that for any states $i, j$,

$$\mathbb{P}\{X_m \neq j,\ m = 0, \dots, n \mid X_0 = i\} \leq C\rho^n.$$

Show that this implies that $\mathbb{E}(T) < \infty$, where $T$ is the first time that the Markov chain reaches the state $j$. (Hint: there exists a $\delta > 0$ such that for all $i$, the probability of reaching $j$ some time in the first $N$ steps, starting at $i$, is greater than $\delta$. Why?)

**1.21** An elementary theorem in number theory states that if two integers $m$ and $n$ are relatively prime (i.e., greatest common divisor equal to 1), then there exist integers $x$ and $y$ (positive or negative) such that

$$mx + ny = 1.$$

Using this theorem show the following:

(a) If $m$ and $n$ are relatively prime then the set

$$\{xm + ny : x, y \text{ positive integers }\}$$

contains all but a finite number of the positive integers.

(b) Let $J$ be a set of nonnegative integers whose greatest common divisor is $d$. Suppose also that $J$ is closed under addition, $m, n \in J \Rightarrow m + n \in J$. Then $J$ contains all but a finite number of integers in the set $\{0, d, 2d, \dots\}$.

# Answers to Week one's Assignments

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## Homework 1: Ordinary differetinal/difference equations

### Question 1

$$\begin{cases} x'(t) = y(t) - x(t) \\ y'(t) = 3x(t) - 3y(t) \end{cases} \implies 3x'(t) = -y'(t).$$

We now take the integral with respect to $t$ from the both sides:

$$3x(t) = -y(t) + c \xrightarrow{x(0)=y(0)=\frac{1}{2}} c = 2.$$

Hence,

$$x(t) = -\frac{1}{3}y(t) + \frac{2}{3}.$$

We substitute $x(t)$ into the original equations in the question:

$$y'(t) = -y(t) + 2 - 3y(t) = -4y(t) + 2.$$

To make the equation homogeneous, we guess $y(t) = at + b$ is a solution. Then,

$$a + 4at + 4b = 2 \Rightarrow a = 0, b = \frac{1}{2}.$$

Now we solve the homogeneous version, where $y'(t) + 4y(t) = 0$ and conclude that $y(t) = c_1 f(t) + \frac{1}{2}$. To find $f(t) = e^{\lambda t}$, we have:

$$\lambda + 4 = 0 \Rightarrow \lambda = -4.$$

Hence,

$$y(t) = c_1 e^{-4t} + \frac{1}{2} \xrightarrow{y(0)=\frac{1}{2}} c_1 = 0.$$

Hence,

$$\begin{cases} y(t) = \frac{1}{2} \\ x(t) = -\frac{1}{3}y(t) + \frac{2}{3} = \frac{1}{2}. \end{cases}$$

### Question 3

We replace $\alpha^n$ in the equation:

$$F_n = F_{n-1} + F_{n-2} \Rightarrow \alpha^2 = \alpha + 1 \Rightarrow \alpha = \frac{1 \pm \sqrt{5}}{2}.$$

Hence,

$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We apply the initial conditions that $F_1 = F_2 = 1$:

$$\begin{cases} c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \\ c_1 \left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 \end{cases} \Rightarrow$$

$$c_2 \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^2 = \left(\frac{1+\sqrt{5}}{2}\right) - 1 \Rightarrow c_2 = -1 \Rightarrow c_1 = \frac{3-\sqrt{5}}{1+\sqrt{5}}.$$

So,

$$F_n = \frac{3-\sqrt{5}}{1+\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

## Question 4

We replace $f(n)$ with $\alpha^n$. Then,

$$3f(n) = f(n-1) + f(n+1) + f(n+2) \Rightarrow \alpha^3 + \alpha^2 - 3\alpha + 1 = 0.$$

$\alpha = 1$ is an answer. Hence,

$$\alpha^3 + \alpha^2 - 3\alpha + 1 = 0 \Rightarrow (\alpha - 1)(\alpha^2 + 2\alpha - 1) = 0.$$

Hence,

$$(\alpha - 1)(\alpha^2 + 2\alpha - 1) = 0 \Rightarrow \begin{cases} \alpha - 1 = 0, \\ \text{or} \\ \alpha^2 + 2\alpha - 1 = 0 \Rightarrow \alpha = -1 \pm \sqrt{2}. \end{cases}$$

Therefore,

$$f(n) = c_1(1)^n + c_2\left(-1-\sqrt{2}\right)^n + c_3\left(\sqrt{2}-1\right)^n.$$

Since $\lim_{n\to\infty} f(n) = 1$: $c_2$ must be zero otherwise $\left(-1-\sqrt{2}\right)^n$ will diverge, $c_3$ would not have an impact since $\left(\sqrt{2}-1\right)^n$ converges to zero and consequently, $c_1 = 1$. On the other hand, $f(0) = 0$, thus $c_1 + c_2 + c_3 = 0 \Rightarrow +1 + 0 + c_3 = 0 \Rightarrow c_3 = -1$. Thus,

$$f(n) = 1 - \left(\sqrt{2}-1\right)^n.$$

## Question 5

First we show that $f(n) = n^2$ is a solution:

$$n^2 = 0.5(n+1)^2 + 0.5(n-1)^2 - 1 = n^2.$$

Now, we find the homogeneous solution:

$$2\alpha^n = \alpha^{n+1} + \alpha^{n-1} \Rightarrow (\alpha - 1)^2 = 0 \Rightarrow \alpha = 1, 1.$$

Hence,

$$f(n) = c_1(1)^n + c_2 n(1)^n + n^2 = c_1 + c_2 n + n^2.$$

## Question 6

### Part a

We replace $f(x) = e^{\lambda x}$ in the ODE. Then, we have:

$$\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm j\sqrt{3}}{2},$$

where $j$ indicates the imaginary number, i.e., $j = \sqrt{-1}$. Hence,

$$f(x) = c_1 e^{\frac{-1+j\sqrt{3}}{2}x} + c_2 e^{\frac{-1-j\sqrt{3}}{2}x}.$$

On the other hand, we know that for $a \in \mathbb{R}$, $e^{jax} = \cos(ax) + j\sin(ax)$. Therefore,

$$f(x) = c_1 e^{-\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) + j\sin\left(\frac{\sqrt{3}}{2}x\right)\right) + c_2 e^{-\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) - j\sin\left(\frac{\sqrt{3}}{2}x\right)\right).$$

To have a real-valued function, we need to make imaginary parts cancel out, hence it must be that $c_1 = c_2$. Then,

$$f(x) = c_1 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right).$$

### Part b

We replace $f(n) = \alpha^n$ in the difference equation and simplify it. Then,

$$\alpha^2 + \alpha + 1 = 0 \Rightarrow \alpha = \frac{-1 \pm j\sqrt{3}}{2}.$$

Hence,

$$f(n) = c_1\left(\frac{-1+j\sqrt{3}}{2}\right)^n + c_2\left(\frac{-1-j\sqrt{3}}{2}\right)^n.$$

With the help of Google Gemini, we realized that the De Moivre's formula asserts for $a \in \mathbb{R}$:

$$(\cos(ax) + j\sin(ax))^n = \cos(anx) + j\sin(anx).$$

Therefore,

$$\begin{aligned} f(n) = c_1\left(\frac{-1+j\sqrt{3}}{2}\right)^n + c_2\left(\frac{-1-j\sqrt{3}}{2}\right)^n &= c_1\left(\cos\frac{4\pi}{6} + j\sin\frac{4\pi}{6}\right)^n + c_2\left(\cos\frac{8\pi}{6} + j\sin\frac{8\pi}{6}\right)^n \\ &= c_1\cos\frac{4n\pi}{6} + jc_1\sin\frac{4n\pi}{6} + c_2\cos\frac{8n\pi}{6} + jc_2\sin\frac{8n\pi}{6}. \end{aligned}$$

Also, for all $n \in \{1, 2, \dots\}$, $\sin\frac{4n\pi}{6} = -\sin\frac{8n\pi}{6}$. Hence, to have a real-valued function we must have that $c_1 = c_2$. Consequently,

$$f(n) = c_1\left(\cos\frac{4n\pi}{6} + \cos\frac{8n\pi}{6}\right).$$

## Homework 2: Markov chains

## Question 1

Yes, it is possible to model this problem as a Markov chain. The state space $S = \{0, 1, 2, 3, 4\}$ is number of papers in the evening after each possible disposal. Then, the transition matrix is as follows:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 1/3 & 0 & 0 & 0 & 2/3 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

## Question 2

We computed $P^3$ using Python software:

$$P^3 = \begin{pmatrix} 0.49537037 & 0.50462963 \\ 0.56770833 & 0.43229167 \end{pmatrix}$$

So, the probability that the chain is in state 1 in time step 3 is 0.50462963,

## Question 3

We use Python software for the computations.

**High powers of $P$.**

$$P^\infty \approx P^{100} = \begin{pmatrix} 0.37878788 & 0.25757576 & 0.36363636 \\ 0.37878788 & 0.25757576 & 0.36363636 \\ 0.37878788 & 0.25757576 & 0.36363636 \end{pmatrix}$$

So, with probability 0.37878788, then chain in the state 1.

**Left eigenvector of $P$.**

$$\bar{\pi} = \bar{\pi}P \Rightarrow \begin{cases} \bar{\pi}_1 = 0.4\bar{\pi}_1 + 0.6\bar{\pi}_2 + 0.2\bar{\pi}_3 \\ \bar{\pi}_2 = 0.2\bar{\pi}_1 + 0.5\bar{\pi}_3 \\ 1 = \bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 \end{cases} \Rightarrow \bar{\pi} = \begin{pmatrix} 0.37878788 \\ 0.25757576 \\ 0.36363636 \end{pmatrix}.$$

So again, with probability 0.37878788, then chain is in state 1.

## Question 5

The communication classes are: $\{0,1\}, \{2,4\}$, and$\{3,5\}$. $\{0,1\}, \{2,4\}$ are recurrent, and $\{3,5\}$ is transient. To compute the probability of being in state 0 in a long run, w compute $P^\infty \approx P^{100}$ using Python:

$$P^{100} = \begin{pmatrix} 0.375 & 0.625 & 0 & 0 & 0 & 0 \\ 0.375 & 0.625 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4375 & 0 & 0.5625 & 0 \\ 0.2386 & 0.3977 & 0.159 & 0 & 0.2046 & 0 \\ 0 & 0 & 0.4375 & 0 & 0.5625 & 0 \\ 0.2046 & 0.3408 & 0.1989 & 0 & 0.2556 & 0 \end{pmatrix}$$

So, with probability 0.375 the chain would be in state 0. Since state 5 is transient, if the chain starts in state 5, again with probability 0.375 the chain will be in state 0 in a long run because, the system would eventually leave state 5 and never comes back to it.

## Question 7

Since the chain is irreducible, for any two states $i, j$ there exits a natural number $N$, where there exits $k \in \{1, \dots N\}$ such that $P^k(i,j) > 0$, meaning that staring from $i$, $j$ will be reached at most in $N$ steps. Let $\delta$ be the minimum probability of reaching any state $j$ from state $i$ in at most $N$ steps ($\delta$ would be nonzero due to our initial explanation regarding irreducibility of the chain). Then, the probability that $j$ is not reached within the first $N$ steps from $i$ would at most be $1-\delta$. To compute the probability that $j$ is not reached within the $n$ steps from $i$, we need to compute how many $N$ steps fit into $n$. This number would be equal to $\lfloor \frac{n}{N} \rfloor$. Then, the probability that $j$ is not reached within the $n$ steps from $i$ is at most $(1-\delta)^{\lfloor \frac{n}{N} \rfloor}$. Now we have:

$$\mathbb{P}\left(X_m \neq j, m = 0, \dots, n \mid X_0 = i\right) \leq (1-\delta)^{\lfloor \frac{n}{N} \rfloor} \leq C\rho^n,$$

where $\rho = (1-\delta)^{\frac{1}{N}} < 1$, and $C = (1-\delta)^{-1} < \infty$ (to account for the floor function and since $1-\delta < 1$). Now, we have:

$$\mathbb{E}[T] = \sum_{n=1}^{\infty} \mathbb{P}(T \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(X_0 \neq j, \dots X_{n-1} \neq j, | X_0 = i) \leq \sum_{n=1}^{\infty} C\rho^n \leq \frac{C}{1-\rho} - C \leq \frac{C}{1-\rho} < \infty.$$

# Question 21

For this question, we obtained assistance from Google Gemini model.

## Part a

First, note that if we want to contain only positive integers, then $n$ and $m$ must be nonnegative. If one of them is negative and the other is positive, we would inevitably contain the whole integers. Also, to rule out the trivial case, let us assume $n, m \geq 1$.

Since $\mathsf{gcd}(n, m) = 1$, the great common divisor of any multiples of $n$ from $1n$ to $mn$ with $m$ is also one. In other words, the set $\{n, 2n, \dots, mn\}$ forms the residue system modulo $m$. Let $N$ be any nonzero integer, we have $N \equiv ny \pmod{m}$, where $1 \leq y \leq m$. Equivalently, we can say $N - ny = mx$ for some integer $x$.

We know that since $1 \leq y \leq m$, $y$ is positive. On the other hand, if $N \geq mn+1$, then $x = \frac{N-ny}{m} > 0$, hence $x$ is also positive. Therefore, we just proved that for positive integers $x$ and $y$, all positive integers bigger than $mn$ are in the set described in the question.

## Part b

Consider the set $J' = \left\{\frac{j}{d} \mid j \in J\right\}$. If we show for any integer $k$ in $J'$ that is bigger than some threshold $K$, all but a finite number of nonnegative integers are in $J'$, then we can conclude for any integer $k$ in $J$ that is bigger than $Kd$, all but a finite number of nonnegative integers are in $J$

By construction, the greatest common divisor of $J'$s elements is one and also $J'$ is closed under addition. Now, consider an arbitrary element in $J'$ and let us denote it by $a$. The set of remainder of $J'$s elements when divided by $a$ is $R = \{j' \pmod{a} \mid j' \in J'\}$. Since the greatest common divisor of $J'$ is one, the greatest common divisor of $R$ is also one. Therefore, $R$ forms the residue system modulo $a$, i.e., $R = \{0, 1, \dots a-1\}$, where for each element of $r \in R$ there exits at least one element in $J'$ denoted by $j'_r$ such that $j'_r \equiv r \pmod{a}$. Let us denote the largest representative of these elements by $K$:

$$K = \max\left\{j'_0, j'_1, \dots j'_{a-1}\right\}.$$

Now we show that any integer bigger than $K$ is in $J'$. Let $k$ be an integer bigger than $K$ and let its remainder when divided by $a$ be $r$, i.e., $k = r \pmod{a}$. We know that there is an element in $J'$ with same remainder, i.e., $j'_r$, therefore:

$$k = j'_r + c \cdot a,$$

for some integer $c$. Since $k > M$ and $j'_r \leq M$, $k - j'_r > 0$ and $c \cdot a$ is positive multiple of $a$. Since $j'_r, a \in J'$, and $J'$ is closed under addition, both $j'_r$ and $c \cdot a = \underbrace{a + a + \cdots + a}_{c \text{ times}}$ are in $J'$. Therefore, their sum, $k$, must also be in $J'$. This proves that $J'$ contains all integers greater than $K$. Consequently, $J$ contains $\{(K+1)d, (K+2)d, \dots\}$, which completes the proof.

**1.9** Consider the Markov chain with state space $\{1, 2, 3, 4, 5\}$ and matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

(a) Is the chain irreducible?
(b) What is the period of the chain?
(c) What are $p_{1,000}(2, 1), p_{1,000}(2, 2), p_{1,000}(2, 4)$ (approximately)?
(d) Let $T$ be the first return time to the state 1, starting at state 1. What is the distribution of $T$ and what is $\mathbb{E}(T)$? What does this say, without any further calculation, about $\pi(1)$?
(e) Find the invariant probability $\bar{\pi}$. Use this to find the expected return time to state 2, starting in state 2.

**1.10** Suppose $X_n$ is a Markov chain with state space $\{0, 1, \ldots, 6\}$ and transition probabilities

$$p(0, 0) = \frac{3}{4}, \quad p(0, 1) = \frac{1}{4},$$

$$p(1, 0) = \frac{1}{2}, \quad p(1, 1) = \frac{1}{4}, \quad p(1, 2) = \frac{1}{4},$$

$$p(6, 0) = \frac{1}{4}, \quad p(6, 5) = \frac{1}{4}, \quad p(6, 6) = \frac{1}{2},$$

and for $j = 2, 3, 4, 5$,

$$p(j, 0) = p(j, j-1) = p(j, j) = p(j, j+1) = \frac{1}{4}.$$

(a) Is this chain irreducible? Is it aperiodic?

(b) Suppose the chain has been running for a long time and we start watching the chain. What is the probability that the next three states will be $4, 5, 0$ in that order?

(c) Suppose the chain starts in state 1. What is the probability that it reaches state 6 before reaching state 0?

(d) Suppose the chain starts in state 3. What is the expected number of steps until the chain is in state 3 again?

(e) Suppose the chain starts in state 0. What is the expected number of steps until the chain is in state 6?

**1.11** Let $X_1, X_2, \ldots$ be the successive values from independent rolls of a standard six-sided die. Let $S_n = X_1 + \cdots + X_n$. Let

$$T_1 = \min\{n \geq 1 : S_n \text{ is divisible by } 8\},$$

$$T_2 = \min\{n \geq 1 : S_n - 1 \text{ is divisible by } 8\}.$$

Find $\mathbb{E}(T_1)$ and $\mathbb{E}(T_2)$. (Hint: consider the remainder of $S_n$ after division by 8 as a Markov chain.)

**1.12** Let $X_n, Y_n$ be independent Markov chains with state space $\{0, 1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \end{array}.$$

Suppose $X_0 = 0, Y_0 = 2$ and let

$$T = \inf\{n : X_n = Y_n\}.$$

(a) Find $\mathbb{E}(T)$.

(b) What is $\mathbb{P}\{X_T = 2\}$?

(c) In the long run, what percentage of the time are both chains in the same state?

[Hint: consider the nine-state Markov chain $Z_n = (X_n, Y_n)$.]

**1.14** Let $X_n$ be a Markov chain on state space $\{1, 2, 3, 4, 5\}$ with transition matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} \end{array}.$$

(a) Is this chain irreducible? Is it aperiodic?

(b) Find the stationary probability vector.

(c) Suppose the chain starts in state 1. What is the expected number of steps until it is in state 1 again?

(d) Again, suppose $X_0 = 1$. What is the expected number of steps until the chain is in state 4?

(e) Again, suppose $X_0 = 1$. What is the probability that the chain will enter state 5 before it enters state 3?

# Solutions to Week two's Assignments

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## Finite Markov chains

### Question 9

#### Part a

Yes, the chain is irreducible because there is only one communicating class.

#### Part b

The period of the chain is three because any state is only visited in multiples of three.

#### Part c

Due to the structure of the chain at no time step there would be a probability for transitioning from state 2 to state 1, nor from state 2 to itself. Hence, $P_{1000}(2,1) = P_{1000}(2,2) = 0$

On the other hand to compute $P_{1000}(2,4)$, since the chain is periodic with the period of three and the remainder of 1000 when divided by 3 is one, the same as number 4, using Python software, we raise the matrix to the power of 4. Hence, $P_{1000}(2,4) \approx P_4(2,4) \approx 0.4167$.

$$P_4 = \begin{pmatrix} 0 & 0.3333 & 0.6665 & 0 & 0 \\ 0 & 0 & 0 & 0.4167 & 0.5835 \\ 0 & 0 & 0 & 0.4167 & 0.5835 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

#### Part d

$T$ is equal to three deterministically, hence $\mathbb{E}[T] = 3$. It says if we compute $\bar{\pi} = \lim_{n\to\infty} \frac{\bar{\phi}P_n + \bar{\phi}P_{n+1} + \bar{\phi}P_{n+2}}{3}$ for some initial state distribution $\bar{\phi}$, then $\bar{\pi}(1) = \frac{1}{3}$.

#### Part e

To find the invariant probability, since the period of the chain is three, using Python software, we compute $P_4, P_5, P_6$ and by averaging them we get $\bar{\pi}$, which would give us $\mathbb{E}[T_2]$.

$$P_4 = \begin{pmatrix} 0 & 0.3333 & 0.6665 & 0 & 0 \\ 0 & 0 & 0 & 0.4167 & 0.5835 \\ 0 & 0 & 0 & 0.4167 & 0.5835 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P_5 = \begin{pmatrix} 0 & 0 & 0 & 0.4167 & 0.5835 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.3333 & 0.6665 & 0 & 0 \\ 0 & 0.3333 & 0.6665 & 0 & 0 \end{pmatrix}$$

$$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.3333 & 0.6665 & 0 & 0 \\ 0 & 0.3333 & 0.6665 & 0 & 0 \\ 0 & 0 & 0 & 0.4167 & 0.5835 \\ 0 & 0 & 0 & 0.4167 & 0.5835 \end{pmatrix}$$

Hence,

$$P_\infty = \frac{P_4 + P_5 + P_6}{3} = \begin{pmatrix} 0.3333 & 0.1111 & 0.2222 & 0.1389 & 0.1945 \\ 0.3333 & 0.1111 & 0.2222 & 0.1389 & 0.1945 \\ 0.3333 & 0.1111 & 0.2222 & 0.1389 & 0.1945 \\ 0.3333 & 0.1111 & 0.2222 & 0.1389 & 0.1945 \\ 0.3333 & 0.1111 & 0.2222 & 0.1389 & 0.1945 \end{pmatrix}.$$

Hence, $\mathbb{E}[T_2] = \frac{1}{\bar{\pi}(2)} = \frac{1}{0.111} = 9$.

## Question 10

### Part a

Yes, the chain is irreducible since there is only one communicating class. Since there are self-loops, the period of the chain is one and thus it's aperiodic.

### Part b

We need to compute the stationary distribution and the probability of being in state four. After that, using the structure of the chain, we can compute how the chain transitions to state five and zero sequentially. Using Python software, we have:

$$P_\infty = \begin{pmatrix} 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \\ 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \\ 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \\ 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \\ 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \\ 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \\ 0.618 & 0.2361 & 0.0902 & 0.0345 & 0.01326 & 0.005306 & 0.002653 \end{pmatrix}.$$

Hence, $\bar{\pi}(4) = 0.01326$. So, the probability that $4, 5, 0$ happens in a long-run is equal to: $0.01326 * \frac{1}{4} * \frac{1}{4} \approx 8.28 \times 10^{-4}$.

### Part c

For this part we obtained assistance from ChatGPT model.

We need to set up a recursive formula connecting how the chain can *hit* state 6 before state 0, while starting at state 1.

Let $h(i)$ be the probability that the chain hits state 6 before state 0, when the chain has started at state $i$. We have that $h(6) = 1$ and $h(0) = 0$. For all other state we have that:

$$h(i) = \sum_{j=1}^{5} P(i,j)h(j).$$

This gives a set of 5 equations with 5 unknowns ($h(1), h(2), h(3), h(4)$, and $h(5)$) as follows:

$$\begin{aligned} h(5) &= \frac{1}{4}h(4) + \frac{1}{4}h(5) + \frac{1}{4}h(6) = \frac{1}{4}h(4) + \frac{1}{4}h(5) + \frac{1}{4} \cdot 1 \\ h(4) &= \frac{1}{4}h(3) + \frac{1}{4}h(4) + \frac{1}{4}h(5) \\ h(3) &= \frac{1}{4}h(2) + \frac{1}{4}h(3) + \frac{1}{4}h(4) \\ h(2) &= \frac{1}{4}h(1) + \frac{1}{4}h(2) + \frac{1}{4}h(3) \\ h(1) &= \frac{1}{2}h(0) + \frac{1}{4}h(1) + \frac{1}{4}h(2) = \frac{1}{2} \cdot 0 + \frac{1}{4}h(1) + \frac{1}{4}h(2). \end{aligned}$$

Solving this system of equations gives us: $h(1) = \frac{1}{144}$.

### Part d

Using $P_\infty$ and the relationship $\mathbb{E}[T_i] = \frac{1}{\bar{\pi}(i)}$, we have:

$$\mathbb{E}[T_3] = \frac{1}{\bar{\pi}(3)} = \frac{1}{0.0345} \approx 29.$$

### Part e

We need to set up a recursive formula connecting the expected number of steps until reaching state 6 for each distinct state.

Let $g(i)$ be the expected number of steps until reaching state 6, when the chain has started at state $i$. We have that $g(6) = 0$. For all other state we have that:

$$g(i) = 1 + \sum_{j=0}^{5} P(i,j)g(j).$$

This gives a set of 6 equations with 6 unknowns $(g(0), g(1), g(2), g(3), g(4), \text{and}\, g(5))$ as follows:

$$\begin{aligned}
g(5) &= 1 + \frac{1}{4}g(0) + \frac{1}{4}g(4) + \frac{1}{4}g(5) + \frac{1}{4}g(6) = 1 + \frac{1}{4}g(0) + \frac{1}{4}g(4) + \frac{1}{4}g(5) + \frac{1}{4}\cdot 0 \\
g(4) &= 1 + \frac{1}{4}g(0) + \frac{1}{4}g(3) + \frac{1}{4}g(4) + \frac{1}{4}g(5) \\
g(3) &= 1 + \frac{1}{4}g(0) + \frac{1}{4}g(2) + \frac{1}{4}g(3) + \frac{1}{4}g(4) \\
g(2) &= 1 + \frac{1}{4}g(0) + \frac{1}{4}g(1) + \frac{1}{4}g(2) + \frac{1}{4}g(3) \\
g(1) &= 1 + \frac{1}{2}g(0) + \frac{1}{4}g(1) + \frac{1}{4}g(2) \\
g(0) &= 1 + \frac{3}{4}g(0) + \frac{1}{4}g(1).
\end{aligned}$$

Solving this system of equations gives us: $g(0) = 928$.

## Question 11

For this question we obtained assistance from Google Gemini model.

The state space of the chain comprises $\{0,1,2,3,4,5,6,7\}$. Let $g(i)$ denotes the expected number of rolls until the chain reaches state zero, while starting at state $i$. We have that $g(0) = 0$. Definitely we won't reach the state 0 with just one roll because the outcome would be among 1 and 6. Hence,

$$\mathbb{E}[T_1] = 1 + \frac{1}{6}\sum_{i=1}^{6} g(i) = 1 + \frac{1}{6}\left(g(1) + g(2) + g(3) + g(4) + g(5) + g(6)\right).$$

On the other hand,

$$\sum_{i=0}^{7} g(i) \overset{g(0)=0}{=} \sum_{i=1}^{7} g(i) = \sum_{i=1}^{7}\left(1 + \frac{1}{6}\sum_{j=1}^{6} g\left(i + j \bmod 8\right)\right).$$

That is, the expected number of rolls starting from state $i \neq 0$ is equal to rolling the die and the expected number of rolls of the resulting state. We rearrange the summations to get:

$$\sum_{i=1}^{7} g(i) = 7 + \frac{1}{6}\sum_{j=1}^{6}\sum_{i=1}^{7} g(i + j \bmod 8).$$

Now for each fixed $j$, $(i+j \bmod 8)$ is a permutation of $\{0,1,2,3,4,5,6,7\}$ excluding the number $j$. Hence, by denoting $S \coloneqq \sum_{i=1}^{7} g(i)$, we have:

$$S = 7 + \frac{1}{6}\sum_{j=1}^{6}(S - g(j)) \Rightarrow \sum_{j=1}^{6} g(j) = 42.$$

Therefore,

$$\mathbb{E}[T_1] = 1 + \frac{1}{6}\sum_{i=1}^{6} g(i) = 1 + \frac{1}{6}\left(g(1) + g(2) + g(3) + g(4) + g(5) + g(6)\right) = 8.$$

For computing $\mathbb{E}[T_2]$, we only need to roll the die once, because the sum of the random variables before the first roll is zero, so $S_{1-1} = S_0 = 0$, and 0 is divisible by 8.

## Question 12

### Part a

We need to set up a recursive formula connecting the expected number of steps until two chains meet.

Let $g(i,j)$ be the expected number of steps until the meeting time, when $\{X_n\}$ has started at state $i$ and $\{Y_n\}$ has started in state $j$. We have that $g(k,k) = 0$, $k \in \{0,1,2\}$. For all other state, since $\{X_n\}$ and $\{Y_n\}$ are independent, we have that:

$$g(i,j) = 1 + \sum_{k=0}^{2}\sum_{l=0}^{2} P(i,k)P(j,l)g(k,j).$$

This gives a set of 6 equations with 6 unknowns $(g(0,1), g(0,2), g(1,0), g(1,2), g(2,0), \text{ and } g(2,1))$ as follows:

$$\begin{aligned}
g(0,1) &= 1 + \frac{1}{2}g(0,0) + \frac{1}{4}g(0,1) + \frac{1}{4}g(0,2)\\
g(0,2) &= 1 + \frac{1}{2}g(0,0) + \frac{1}{4}g(0,1) + \frac{1}{4}g(0,2)\\
g(1,0) &= 1 + \frac{1}{4}g(1,0) + \frac{1}{4}g(1,1) + \frac{1}{2}g(1,2)\\
g(1,2) &= 1 + \frac{1}{4}g(1,0) + \frac{1}{4}g(1,1) + \frac{1}{2}g(1,2)\\
g(2,0) &= 1 + 0 \cdot g(2,0) + \frac{1}{2}g(2,1) + \frac{1}{2}g(2,2)\\
g(2,1) &= 1 + 0 \cdot g(2,0) + \frac{1}{2}g(2,1) + \frac{1}{2}g(2,2).
\end{aligned}$$

Solving this system of equations gives us: $g(0,2) = \frac{118}{35} \approx 3.37$.

### Part b

We need to set up a recursive formula connecting how the chains can meet at state 2 while starting at state 0 and 2 respectively.

Let $h(i,j)$ be the probability of first meeting at state 2, when $\{X_n\}$ has started at state $i$ and $\{Y_n\}$ has started in state $j$. We have that $h(1,1) = h(0,0) = 0$ (first meeting should be at state 2), and $h(2,2) = 1$. For all other state, since $\{X_n\}$ and $\{Y_n\}$ are independent, we have that:

$$h(i,j) = \sum_{k=0}^{2}\sum_{l=0}^{2} P(i,k)P(j,l)h(k,j).$$

This gives a set of 6 equations with 6 unknowns $(h(0,1), h(0,2), h(1,0), h(1,2), h(2,0),$ and $h(2,1))$ as:

$$\begin{aligned}
h(0,1) &= \frac{1}{2}h(0,0) + \frac{1}{4}h(0,1) + \frac{1}{4}h(0,2) \\
h(0,2) &= \frac{1}{2}h(0,0) + \frac{1}{4}h(0,1) + \frac{1}{4}h(0,2) \\
h(1,0) &= \frac{1}{4}h(1,0) + \frac{1}{4}h(1,1) + \frac{1}{2}h(1,2) \\
h(1,2) &= \frac{1}{4}h(1,0) + \frac{1}{4}h(1,1) + \frac{1}{2}h(1,2) \\
h(2,0) &= 0 \cdot h(2,0) + \frac{1}{2}h(2,1) + \frac{1}{2}h(2,2) \\
h(2,1) &= 0 \cdot h(2,0) + \frac{1}{2}h(2,1) + \frac{1}{2}h(2,2).
\end{aligned}$$

Solving this system of equations gives us: $h(0,2) = \frac{15}{28} \approx 0.535$.

### Part c

Since the chains are independent, we need to multiple their invariant distributions for states $\{1,2,3\}$ and sum them up. Let us find the invariant distribution by solving $\bar{\pi} = \bar{\pi}P$:

$$\begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \\ \bar{\pi}_3 \end{pmatrix} = \begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \\ \bar{\pi}_3 \end{pmatrix}^\top \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \quad \text{and} \quad \bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 = 1.$$

The solution of this equation is: $\bar{\pi} = \frac{1}{11}(2,4,5)$. Hence, the probability that both chains spend time at the same states in a long run is equal to: $\frac{1}{11^2}(4+16+25) = \frac{45}{121}$.

## Question 14

### Part a

Yes, the chain irreducible since there is only one communicating class. Since state 5 has a self-loop, the period of the chain is one and the chain is aperiodic.

### Part b

We should solve:

$$\begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \\ \bar{\pi}_3 \\ \bar{\pi}_4 \\ \bar{\pi}_5 \end{pmatrix} = \begin{pmatrix} \bar{\pi}_1 \\ \bar{\pi}_2 \\ \bar{\pi}_3 \\ \bar{\pi}_4 \\ \bar{\pi}_5 \end{pmatrix}^\top \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix} \quad \text{and} \quad \bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 + \bar{\pi}_4 + \bar{\pi}_5 = 1.$$

The solution of this equation is: $\bar{\pi} = \frac{1}{37}(10,5,5,3,14)$.

### Part c

$$\mathbb{E}[T_1] = \frac{1}{\bar{\pi}(1)} = 3.7.$$

### Part d

We need to set up a recursive formula connecting the expected number of steps until reaching state 4 for each distinct state. Let $g(i)$ be the expected number of steps until reaching state 4, when the chain has started at state $i$. We have that $g(4) = 0$. For all other state we have that:

$$g(i) = 1 + \sum_{j=0, j\neq 4}^{5} P(i,j)g(j).$$

This gives a set of 4 equations with 4 unknowns $(g(1), g(2), g(3), \text{ and } g(5))$ as follows:

$$\begin{aligned} g(5) &= 1 + \frac{1}{2}g(0) + \frac{1}{2}g(5) \\ g(3) &= 1 + \frac{2}{5}g(4) + \frac{3}{5}g(5) \\ g(2) &= 1 + \frac{1}{5}g(4) + \frac{4}{5}g(5) \\ g(1) &= 1 + \frac{1}{2}g(2) + \frac{1}{2}g(3) \end{aligned}$$

Solving this system of equations gives us: $g(1) = \frac{34}{3}$.

### Part e

We need to set up a recursive formula connecting how the chain can *hit* state 5 before state 3, while starting at state 1.

Let $h(i)$ be the probability that the chain hits state 5 before state 3, when the chain has started at state $i$. We have that $h(5) = 1$ and $h(3) = 0$. For all other state we have that:

$$h(i) = \sum_{j=1, j\neq 3}^{4} P(i,j)h(j).$$

This gives a set of 3 equations with 3 unknowns $(h(1), h(2), \text{ and } h(4))$ as follows:

$$\begin{aligned} h(4) &= h(1) \\ h(2) &= \frac{1}{5}h(4) + \frac{4}{5}h(5) = \frac{1}{5}h(4) + \frac{4}{5} \\ h(1) &= \frac{1}{2}h(2) + \frac{1}{2}h(3) = \frac{1}{2}h(2) \end{aligned}$$

Solving this system of equations gives us: $h(1) = \frac{4}{9}$.

**1.15** Let $X_n$ be an irreducible Markov chain with state space $S$ starting at state $i$ with transition matrix $\mathbf{P}$. Let

$$T = \min\{n > 0 : X_n = i\}$$

be the first time that the chain returns to state $i$. For each state $j$ let $r(j)$ be the expected number of visits to $j$ before returning to $i$,

$$r(j) = \mathbb{E}\left[\sum_{n=0}^{T-1} I\{X_n = j\}\right].$$

Note that $r(i) = 1$.

(a) Let $\bar{r}$ be the vector whose $j$th component is $r(j)$. Show that $\bar{r}\mathbf{P} = \bar{r}$.

(b) Show that

$$\mathbb{E}(T) = \sum_{j \in S} r(j).$$

(c) Conclude that $\mathbb{E}(T) = \pi(i)^{-1}$, where $\bar{\pi}$ denotes the invariant probability.

**1.16** Consider simple random walk on the circle $\{0, 1, \ldots, N-1\}$ started at 0 as described in Section 1.6. Show that the distribution of $X_{T_N}$ is uniform on $\{1, 2, \ldots, N-1\}$.

**1.18** Suppose we take a standard deck of cards with 52 cards and do the card shuffling procedure as in Section 1.6. Suppose we do one move every second. What is the expected amount of time in years until the deck returns to the original order?

**1.19** Suppose we flip a fair coin repeatedly until we have flipped four consecutive heads. What is the expected number of flips that are needed? (Hint: consider a Markov chain with state space $\{0, 1, \ldots, 4\}$.)

# Answers to Week three's Assignments

Alireza Kazemipour

December 16, 2025

## Finite Markov chains

### Question 15

#### Part a

$$\begin{aligned}
(\bar{r}P)_j &= \sum_{k\in S} P(k,j)\bar{r}(k) \\
&= \sum_{k\in S} P(k,j)\,\mathbb{E}\left[\sum_{n=0}^{T-1} \mathbb{I}\{X_n = k\}\right] \\
&= \mathbb{E}\left[\sum_{n=0}^{T-1}\sum_{k\in S} P(k,j)\mathbb{I}\{X_n = k\}\right] \\
&= \mathbb{E}\left[\sum_{n=0}^{T-1} P(X_n, j)\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{T} \mathbb{E}\left[X_{n+1} = j \mid X_n\right]\right] \\
&= \sum_{n=0}^{T-1} \mathbb{E}\left[X_{n+1} = j\right] && \text{(tower rule)} \\
&= \sum_{n=1}^{T} \mathbb{E}\left[X_n = j\right] \\
&= \sum_{n=0}^{T-1} \mathbb{E}\left[X_n = j\right] && (X_0 = X_T = i.) \\
&= \mathbb{E}\left[\sum_{n=0}^{T-1} \mathbb{I}\{X_n = j\}\right] \\
&= r(j).
\end{aligned}$$

#### Part b

$$T = \sum_{n=0}^{T-1} 1 = \sum_{n=0}^{T-1}\sum_{j\in S} \mathbb{I}\{X_n = j\} = \sum_{j\in S}\sum_{n=0}^{T-1} \mathbb{I}\{X_n = j\}.$$

We take the expectation from both sides; we have:

$$\mathbb{E}\left[T\right] = \sum_{j\in S} \mathbb{E}\left[\sum_{n=0}^{T-1} \mathbb{I}\{X_n = j\}\right] = \sum_{j\in S} r(j).$$

Taking expectations and changing the order of summations were valid since we are in a finite setting.

### Part c

Define the softmax distribution $\mu(j) = \frac{r(j)}{\mathbb{E}[T]}$, $\forall j \in S$. Since $\bar{r} = \bar{r}P$, then $\mu = \mu P$. Since the invariant probability distribution is unique, then $\bar{\pi} = \mu$, and therefore $\bar{\pi}(i) = \mu(i) = \frac{r(i)}{\mathbb{E}[T]}$, and since $r(i) = 1$, then $\mathbb{E}[T] = \frac{1}{\bar{\pi}(i)}$.

## Question 16

To solve this question, we obtained assistance from Google Gemini model.

The chain starts at zero, hence $\mathbb{P}(X_{T_N} = 0) = 0$. Therefore, we show that for any state $k \in \{1, \dots, N-1\}$, $\mathbb{P}(X_{T_N} = k) = 1/(N-1)$.

Fix two arbitrary states $i, j \in \{1, \dots, N-1\}$ and let $\mathcal{E}_{ij}$ be the event that all states except $i, j$ are visited. So, $i, j$ are the only remaining states to be visited. We have that:

$$\begin{aligned}\mathbb{P}(\text{hitting } i \text{ before } j \mid \mathcal{E}_{ij}) =& 0.5 * \mathbb{P}(\text{being in the neighbor of } i \mid \mathcal{E}_{ij}) + \\ & 0.5 * \mathbb{P}(\text{being in the neighbor of } j \mid \mathcal{E}_{ij}).\end{aligned}$$

Let $i$ be our reference state. Suppose the longer distance between $i$ and $j$ is d. Then,

$$\mathbb{P}(\text{being in the neighbor of } i \mid \mathcal{E}_{ij}) = \frac{1}{d}, \text{ and } \mathbb{P}(\text{being in the neighbor of } j \mid \mathcal{E}_{ij}) = \frac{d-1}{d}.$$

Therefore,

$$\mathbb{P}(\text{hitting } i \text{ before } j \mid \mathcal{E}_{ij}) = 0.5\left(\frac{1}{d} + \frac{d-1}{d}\right) = 0.5.$$

The result is identical if choose the shorter distance between $i, j$ as well.

Hence, we proved that for every two arbitrary state, each of them will be the last visited state with equal probability. Since these two states were chosen arbitrarily, the results are extended to all states in $\{1, \dots, N-1\}$ in which every state has the probability of $\frac{1}{N-1}$ to be the last visited.

## Question 18

$$\frac{\frac{1}{\bar{\pi}(52)}}{\text{seconds per year}} = \frac{52!}{365 * 86400}$$

## Question 19

The transition matrix is equal to:

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We decompose $P$ in the following way:

$$P = \left(\begin{array}{c|c} Q & S \\ \hline \mathbf{0} & 1 \end{array}\right).$$

Hence,

$$Q = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad M = (I - Q)^{-1} = \begin{pmatrix} 16 & 8 & 4 & 2 \\ 14 & 8 & 4 & 2 \\ 12 & 6 & 4 & 2 \\ 8 & 4 & 2 & 2 \end{pmatrix}.$$

The expected number of steps from 0 to 4 is equal to multiplying the vector $(1, 1, 1, 1)$ to the first column of $M$ that corresponds to state 0, Hence,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 16 \\ 14 \\ 12 \\ 8 \end{pmatrix} = 30.$$

**2.2** Consider the following Markov chain with state space $S = \{0, 1, \ldots\}$. A sequence of positive numbers $p_1, p_2, \ldots$ is given with $\sum_{i=1}^{\infty} p_i = 1$. Whenever the chain reaches state 0 it chooses a new state according to the $p_i$. Whenever

the chain is at a state other than 0 it proceeds deterministically, one step at a time, toward 0. In other words, the chain has transition probability

$$p(x, x-1) = 1, \quad x > 0,$$

$$p(0, x) = p_x, \quad x > 0.$$

This is a recurrent chain since the chain keeps returning to 0. Under what conditions on the $p_x$ is the chain positive recurrent? In this case, what is the limiting probability distribution $\pi$? [Hint: it may be easier to compute $\mathbb{E}(T)$ directly where $T$ is the time of first return to 0 starting at 0.]

**2.3** Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition probabilities:

$$p(x, x+1) = 2/3; \quad p(x, 0) = 1/3.$$

Show that the chain is positive recurrent and give the limiting probability $\pi$.

**2.5** Let $X_n$ be the Markov chain with state space $\mathbb{Z}$ and transition probability

$$p(n, n+1) = p, \quad p(n, n-1) = 1 - p,$$

where $p > 1/2$. Assume $X_0 = 0$.

(a) Let $Y = \min\{X_0, X_1, \ldots\}$. What is the distribution of $Y$?

(b) For positive integer $k$, let $T_k = \min\{n : X_n = k\}$ and let $e(k) = \mathbb{E}[T_k]$. Explain why $e(k) = k\, e(1)$.

(c) Find $e(1)$. (Hint: (b) might be helpful.)

(d) Use (c) to give another proof that $e(1) = \infty$ if $p = 1/2$.

**2.6** Suppose $J_1, J_2, \ldots$ are independent random variables with $\mathbb{P}\{J_j = 1\} = 1 - \mathbb{P}\{J_j = 0\} = p$. Let $k$ be a positive integer and let $T_k$ be the first time that $k$ consecutive 1s have appeared. In other words, $T_k = n$ if $J_n = J_{n-1} = \cdots = J_{n-(k-1)} = 1$ and there is no $m < n$ such that $J_m = J_{m-1} = \cdots = J_{m-(k-1)} = 1$. Let $X_0 = 0$ and for $n > 0$, let $X_n$ be the number of consecutive 1s in the last run, i.e., $X_n = k$ if $J_{n-k} = 0$ and $J_i = 1$ for $n - k < i \leq n$.

(a) Explain why $X_n$ is a Markov chain with state space $\{0, 1, 2, \ldots\}$ and give the transition probabilities.

(b) Show that the chain is irreducible and positive recurrent and give the invariant probability $\pi$.

(c) Find $\mathbb{E}[T_k]$ by writing an equation for $\mathbb{E}[T_k]$ in terms of $\mathbb{E}[T_{k-1}]$ and then solving the recursive equation.

(d) Find $\mathbb{E}[T_k]$ is a different way. Suppose the chain starts in state $k$, and let $\hat{T}_k$ be the the time until returning to state $k$ and $\hat{T}_0$ the time until the chain reaches state 0. Explain why

$$\mathbb{E}[\hat{T}_k] = \mathbb{E}[\hat{T}_0] + \mathbb{E}[T_k],$$

find $\mathbb{E}[\hat{T}_0]$, and use part (b) to determine $\mathbb{E}[\hat{T}_k]$.

# Answers to Week Four's Assignments

Alireza Kazemipour

December 16, 2025

## Countable Markov chains

### Question 2

In order to have a positive recurrent chain, we must have $\mathbb{E}[T_i] < \infty$, $\forall i \in S$. So for state 0, we should have:

$$\mathbb{E}[T_0] = \underbrace{1}_{\text{the first move from state 0}} + \sum_i i \cdot p_i < \infty,$$

hence the required condition is that $\sum_i i \cdot p_i < \infty$.

Now we compute the invariant probability distribution $\pi(i) = \sum_j \pi(j)p(j,i)$. For state 0 have:

$$\pi(0) = \pi(1)p(1,0) = \pi(1).$$

for states 1 and 2, we have:

$$\pi(1) = \pi(0)p(0,1) + \pi(2)p(2,1) = \pi(0)p(0,1) + \pi(2) \Rightarrow \pi(2) \overset{\pi(1)=\pi(0)}{=} \pi(0) - \pi(0)p_1,$$
$$\pi(2) = \pi(0)p(0,2) + \pi(3)p(3,2) = \pi(0)p(0,2) + \pi(3) \Rightarrow \pi(3) = \pi(2) - \pi(0)p_2 = \pi(0) - \pi(0)p_1 - \pi(0)p_2.$$

Hence, for any state $i \geq 1$ we have

$$\pi(i) = \pi(0)\left(1 - \sum_{j=1}^{i} p_j\right) = \pi(0)\sum_{j=i}^{\infty} p_j.$$

Now we compute $\pi(0)$ using the fact that $\sum_{i=0}^{\infty} \pi(i) = 1$. Using the above display we have

$$\sum_{i=0}^{\infty}\pi(i) = \pi(0) + \sum_{i=1}^{\infty}\pi(i) = \pi(0)\left(1 + \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} p_j\right) = \pi(0)\left(1 + \sum_{j=1}^{\infty}\sum_{i=1}^{j} p_j\right) = \pi(0)\left(1 + \sum_{j=1}^{\infty} j \cdot p_j\right).$$

Therefore,

$$\pi(0)\left(1 + \sum_{j=1}^{\infty} j \cdot p_j\right) = 1 \Rightarrow \pi(0) = \frac{1}{1 + \sum_{j=1}^{\infty} j \cdot p_j}.$$

And for $i \geq 1$ we have:

$$\pi(i) = \pi(0)\sum_{k=i}^{\infty} p_k \Rightarrow \frac{\sum_{k=i}^{\infty} p_k}{1 + \sum_{j=1}^{\infty} j \cdot p_j}.$$

### Question 3

The chain is irreducible because from any state we can get to state 0 with probability $\frac{1}{3}$, and from state 0 we can get to any other state $k$ with non-zero probability of $(\frac{2}{3})^k$. Hence, to prove that the chain is positive recurrent, it suffices to show that the expected first return time to state 0 is finite and other states would have finite expected first return times as well. Let $g(n)$ denote the expected number of steps that it takes to return to state 0, starting at state $n$. Then,

$$g(n) = \frac{1}{3} \times 1 + \frac{2}{3} \times g(n+1).$$

This is an inhomogeneous difference equation and one solution is $g(n) = 1$. Now we solve the homogeneous version:

$$\lambda^n = \frac{2}{3}\lambda^{n+1},$$

which result in $\lambda = \frac{3}{2}$, and the final solution would be:

$$g(n) = c\left(\frac{3}{2}\right)^n + 1, \quad n \in \{0, 1, \dots\}, c \in \mathbb{R}.$$

$g(0) = c + 1 < \infty$, which completes our claim that the chain is positive recurrent. Now we find the invariant probability distribution $\pi$. For state 0 we have:

$$\pi(0) = \frac{1}{3}\sum_{i \geq 0} \pi(i) = \frac{1}{3} \times 1 = \frac{1}{3}.$$

For states $i \geq 1$, we have:

$$\pi(i) = \frac{2}{3}\pi(i-1).$$

This is a homogeneous difference equation with the solution $\pi(i) = c\left(\frac{2}{3}\right)^i$. To find $c$, we use $\pi(0)$ as follows:

$$\pi(1) = c\frac{2}{3} = \frac{2}{3}\pi(0) = \frac{2}{3} \cdot \frac{1}{3},$$

which gives us $c = \frac{1}{3}$. Hence,

$$\pi(i) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^i, \quad i \geq 0.$$

## Question 5

### Part a

We will show that the distribution of $Y$ is the geometric distribution. The minimum will occur in the set $\{\dots, -2, -1, 0\}$. Hence, we should compute the probability of continuously moving left until a certain point. For any state $i \in \mathbb{Z}$, let

$$\rho = \mathbb{P}\left(\text{the chain hits } i-1 \mid X_0 = i\right).$$

Hence, if the chains moves left in one step then $\rho = (1-p)$ and if the chains moves right, then the chain should first hit $i$ from $i+1$ and then $i-1$ from $i$. But, the probability of hitting $i$ from $i+1$ is the same as hitting $i-1$ from $i$ both equal to $\rho$. Due to independence, the probability of hitting $i-1$ from $i+1$ would be $\rho^2$. Therefore,

$$\rho = (1-p) + p\rho^2.$$

Solving this equation gives us

$$\rho = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm |1-2p|}{2p} \overset{p>1/2}{=} \frac{1 \pm (2p-1)}{2p} = \begin{cases} 1, \\ \text{or} \\ \frac{1-p}{p} \end{cases}.$$

Since, the chain is transient ($p > 1/2$) and moves toward $+\infty$, $\rho$ must be less than 1, thus $\rho = \frac{1-p}{p}$.

Suppose $k \in \{\dots, -1, 0\}$ is the minimum. Starting from 0, it takes $\rho^k$ steps to reach $k$ on the left and with probability $1-\rho$, the chain would never move left from $k$ any further, hence

$$\mathbb{P}(Y = k) = (\rho)^k(1-\rho) = \left(\frac{1-p}{p}\right)^k \cdot \left(\frac{2p-1}{p}\right).$$

### Part b

Since $T_0 = 0$, using telescopic sum we have that $T_k = \sum_{i=1}^{k} T_i - T_{i-1}$. Hence,

$$\mathbb{E}[T_k] = \mathbb{E}\left[\sum_{i=1}^{k}(T_i - T_{i-1})\right] = \sum_{i=1}^{k}\mathbb{E}[T_i - T_{i-1}].$$

Due to independence, the time between the first visits of any consecutive state $i$ and $i-1$ is the same as the time between the first visits of 0 and 1, hence

$$\mathbb{E}[T_k] = \mathbb{E}\left[\sum_{i=1}^{k}(T_i - T_{i-1})\right] = \sum_{i=1}^{k}\mathbb{E}[T_i - T_{i-1}] = \sum_{i=1}^{k}\mathbb{E}[T_1 - T_0] \overset{(T_0=0)}{=} k\,\mathbb{E}[T_1] = ke(1).$$

### Part c

Let $g(i \to 1)$ be the expected time of reaching state 1 starting rom state $i$. We have:

$$g(0 \to 1) = 1 + p * \underbrace{g(1 \to 1)}_{=0} + (1-p)g(-1 \to 1).$$

But due to independence,

$$g(-1 \to 1) = g(-1 \to 0) + g(0 \to 1).$$

And the time it takes to go from state $-1$ to 0 is the same as the time it takes to go from state 0 to 1. Thus,

$$g(-1 \to 1) = g(-1 \to 0) + g(0 \to 1) = 2g(0 \to 1).$$

So we have,

$$g(0 \to 1) = 1 + (1-p)g(-1 \to 1) = p + 2(1-p)g(0 \to 1).$$

From the above display we get

$$e(1) = g(0 \to 1) = \frac{1}{2p-1}.$$

### Part d

Let us assume that if $p = \frac{1}{2}$, then $e(1) < \infty$. We show that this assumption leads to a contradiction. In Part (c) we showed that

$$e(1) = 1 + 2(1-p)e(1) \xrightarrow{p=\frac{1}{2}} e(1) = 1 + e(1) \Rightarrow 0 = 1.$$

$1 = 0$ is a contradiction, hence $e(1) = \infty$.

## Question 6

### Part a

Since at each time step the probability that $J_i = 1$ is independently determined (we have independent Bernoulli trials), the information needed to compute the probability that $X_n$ becomes $k+1$ or 0 only depends on $X_n = k$ (because $X_{n+1}$ can be equal to $k+1$ if $X_n$ is equal to $k$ first), for any positive integer $k$. Hence, the chain is Markov. The transition probabilities:

$$\mathbb{P}(X_{n+1} = k+1 \mid X_n = k) = p, \quad \mathbb{P}(X_{n+1} = 0 \mid X_n = k) = 1-p$$

### Part b

The chain is irreducible because we can get back to state 0 from any state with probability $1-p$ and we can get to any state $k$ from state 0 in $k$ steps with probability $p^k \neq 0$.

Since the chain is irreducible, it suffices to show that $\mathbb{E}[T_0] < \infty$ to conclude that the chain is positive recurrent. Let $g_0(i)$ be the expected number of steps until getting to state 0 for the first time when starting from state $i$. We have that:

$$g_0(0) = 1 + (1-p) * 0 + p * g_0(1)$$
$$g_0(1) = 1 + (1-p) * 0 + p * g_0(2).$$

Hence, we can see the terms of the general formula:

$$\mathbb{E}[T_0] = g_0(0) = \sum_{i=0}^{\infty} p^i = \frac{1}{1-p} < \infty,$$

which completes the claim that the chain is positive recurrent. To find the invariant probability and considering that the chain is positive recurrent, we have:

$$\pi(0) = \frac{1}{g_0(0)} = 1-p, \quad \pi(i) = p \cdot \pi(i-1),\ i \geq 1.$$

By solving the difference equation for $\pi(i)$, we have that $\pi(i) = c \cdot p^i$, $c \in \mathbb{R}$. To find $c$ we use the requirement that the sum of probabilities should be equal to 1:

$$\pi(0) + \sum_{i \geq 1} c \cdot p^i = 1 \Rightarrow c = (1-p).$$

Hence,

$$\pi(i) = (1-p)p^i, \quad i \in \{0, 1, 2, \dots\}.$$

### Part c

Let $E_k := \mathbb{E}[T_k]$. We have:

$$E_k = p(E_{k-1} + 1) + (1-p)(E_{k-1} + 1 + E_k).$$

The above equation says that by reaching state $k-1$ for the first time, we either succeed to reach state $k$ in one step by probability $p$, or the chain goes back to state 0 and we should start all over again. The above equation simplifies to:

$$E_k = \frac{1}{p} E_{k-1} + \frac{1}{p}.$$

By writing a few terms of the above sum:

$$E_0 = 0, \quad E_1 = \frac{1}{p}, \quad E_2 = \frac{1}{p} + \frac{1}{p^2}, \dots,$$

we realize that

$$\mathbb{E}[T_k] = E_k = \sum_{i=1}^{k} \left(\frac{1}{p}\right)^i = \frac{1/p\left(1 - (1/p)^k\right)}{1 - 1/p} = \frac{p^k - 1}{p^k(p-1)}.$$

### Part d

Since each state can only be reached from its predecessor, in order to visit state $k$ more than once, the chain has go back to state 0 first and moves its way back up. Getting back to state 0 takes $\mathbb{E}\left[\hat{T}_0\right]$ steps, and then it takes $\mathbb{E}[T_k]$ to get back state $k$. Hence,

$$\mathbb{E}\left[\hat{T}_k\right] = \mathbb{E}\left[\hat{T}_0\right] + \mathbb{E}[T_k]$$

Since the chain is irreducible and positive recurrent, using part (b) and the fact that $\mathbb{E}\left[\hat{T}_k\right] = \frac{1}{\pi(k)}$ for irreducible positive recurrent chains, we have that

$$\mathbb{E}\left[\hat{T}_0\right] = \frac{1}{\pi(0)} = \frac{1}{1-p}, \quad \mathbb{E}\left[\hat{T}_k\right] = \frac{1}{\pi(k)} = \frac{1}{(1-p)p^k}.$$

Hence,

$$\mathbb{E}\left[T_k\right] = \frac{1}{(1-p)p^k} - \frac{1}{1-p} = \frac{p^k - 1}{p^k(p-1)}.$$

**2.7** Let $X_n$ be a Markov chain with state space $S = \{0, 1, 2, \dots\}$. For each of the following transition probabilities, state if the chain is positive recurrent, null recurrent, or transient. If it is positive recurrent, give the stationary probability distribution:

(a) $p(x, 0) = 1/(x+2), \quad p(x, x+1) = (x+1)/(x+2)$;

(b) $p(x, 0) = (x+1)/(x+2), \quad p(x, x+1) = 1/(x+2)$;

(c) $p(x, 0) = 1/(x^2+2), \quad p(x, x+1) = (x^2+1)/(x^2+2)$.

**2.9** Consider the branching process with offspring distribution as in Exercise 2.8(b) and suppose $X_0 = 1$.

(a) What is the probability that the population is extinct in the second generation ($X_2 = 0$), given that it did not die out in the first generation ($X_1 > 0$)?

(b) What is the probability that the population is extinct in the third generation, given that it was not extinct in the second generation?

positive recurrent, null recurrent, transient.

**2.11** Consider the following variation of the branching process. At each time $n$, each individual produces offspring independently using offspring distribution $\{p_n\}$, and then the individual dies with probability $q \in (0, 1)$. Hence, each individual reproduces $j$ times where $j$ is the lifetime of the individual. For which values of $q$, $\{p_n\}$ do we have eventual extinction with probability one?

**2.15** Let $X_1, X_2, \ldots$ be independent identically distributed random variables taking values in the integers with mean 0. Let $S_0 = 0$ and

$$S_n = X_1 + \cdots + X_n.$$

(a) Let

$$G_n(x) = \mathbb{E}\left[\sum_{j=0}^{n} I\{S_j = x\}\right]$$

be the expected number of visits to $x$ in the first $n$ steps. Show that for all $n$ and $x$, $G_n(0) \geq G_n(x)$. (Hint: consider the first $j$ with $S_j = x$.)

(b) Recall that the law of large numbers implies that for each $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{|S_n| \leq n\epsilon\} = 1.$$

Show that this implies that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{|x|\leq \epsilon n} G_n(x) = 1.$$

(c) Using (a) and (b), show that for each $M < \infty$ there is an $n$ such that $G_n(0) \geq M$.

(d) Conclude that $S_n$ is a recurrent Markov chain.

# Answers to Week Five's Assignments

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## Countable Markov chains

### Question 7

First we note that the chains of each section are irreducible and not transient. Because from any state $x$ we can get to state 0 with non-zero probability and vice versa. Therefore, to determine if chains are null or positive recurrent, it suffices to determine if $\mathbb{E}[T_0]$, the non-zero expected return time to state 0, is finite or infinite. Let $g(i)$ denote the expected number of steps to get to state 0 starting from state $i$. Then, for each section we would have the following:

#### Part a

Intuitively, we expect this chain to be null recurrent because as $x$ tends to infinity, $p(x, x+1)$ tends to one. We formally show that $g(0) = \infty$.

Using the transition probabilities we have:

$$g(0) = 1 + \frac{1}{2} * g(1) + \frac{1}{2} * 0$$
$$g(1) = 1 + \frac{2}{3} * g(2) + \frac{1}{3} * 0$$
$$g(2) = 1 + \frac{3}{4} * g(3) + \frac{1}{4} * 0$$
$$g(3) = 1 + \frac{4}{5} * g(4) + \frac{1}{5} * 0$$

Hence,

$$g(0) = \sum_{k=1}^{\infty} \frac{1}{k}.$$

But, $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series that diverges in the sense that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$. So matching our intuition, the chain is null recurrent.

#### Part b

Intuitively, we expect this chain to be positive recurrent because as $x$ tends to infinity, $p(x, x+1)$ tends to zero. We formally show that $g(0) < \infty$.

Using the transition probabilities we have:

$$g(0) = 1 + \frac{1}{2} * g(1) + \frac{1}{2} * 0$$
$$g(1) = 1 + \frac{1}{3} * g(2) + \frac{2}{3} * 0$$
$$g(2) = 1 + \frac{1}{4} * g(3) + \frac{3}{4} * 0$$
$$g(3) = 1 + \frac{1}{5} * g(4) + \frac{4}{5} * 0$$

Hence,

$$g(0)=\sum_{k=1}^{\infty}\frac{1}{k!}\overset{(e^1 \text{ Taylor's expansion})}{=} e-1<\infty.$$

Hence matching our intuition, the chain is positive recurrent. The above calculation gives us $\pi(0)=\frac{1}{e-1}$. For other states $x\geq 1$ we have:

$$\pi(x)=\frac{1}{x+1}\pi(x-1).$$

The recursive nature of the above formula for $\pi(x)$, leads to

$$\pi(x)=\pi(0)\cdot\Pi_{k=1}^{x}\frac{1}{k+1}=\pi(0)\frac{1}{(x+1)!}=\frac{1}{(1-e)(x+1)!}.$$

### Part c

Intuitively, we expect this chain to be null recurrent because as $x$ tends to infinity, $p(x,x+1)$ tends to one. We formally show that $g(0)=\infty$.

Using the transition probabilities we have:

$$\begin{aligned} g(0)&=1+\frac{1}{2}*g(1)+\frac{1}{2}*0\\ g(1)&=1+\frac{2}{3}*g(2)+\frac{1}{3}*0\\ g(2)&=1+\frac{5}{6}*g(3)+\frac{1}{6}*0\\ g(3)&=1+\frac{10}{11}*g(4)+\frac{1}{11}*0 \end{aligned}$$

Hence,

$$g(0)=1+\sum_{k=0}^{\infty}\Pi_{j=0}^{k}\frac{j^2+1}{j^2+2}=1+\sum_{k=0}^{\infty}\Pi_{j=0}^{k}\left(1-\underbrace{\frac{1}{j^2+2}}_{:=c_j}\right)=1+\sum_{k=0}^{\infty}\Pi_{j=0}^{k}\left(1-c_j\right).$$

We have that

$$\ln\left(\Pi_{j=0}^{k}(1-c_j)\right)=\sum_{j=0}^{k}\ln(1-c_j)\overset{(\text{Taylor's expansion})}{\approx}-\sum_{j=0}^{k}c_j.$$

As $k$ tends to infinity, since $\sum_{j=0}^{\infty}c_j=\sum_{j=0}^{\infty}\frac{1}{j^2+2}\leq\frac{1}{2}+\sum_{j=1}^{\infty}\frac{1}{j^2}$, and $\sum_{j=1}^{\infty}\frac{1}{j^2}$ is convergent, $\sum_{j=0}^{\infty}c_j=\sum_{j=0}^{\infty}\frac{1}{j^2+2}$ is also convergent. Therefore, the term $\Pi_{j=0}^{\infty}(1-c_j)$ converges to a positive real number $L$ (smaller than one). Thus, there exists a $K\geq 0$, where for all $k\geq K$, $\sum_{k=K}^{\infty}\Pi_{j=0}^{k}\left(1-c_j\right)\approx\sum_{k=K}^{\infty}L=\infty$. Since

$$g(0)=1+\sum_{k=0}^{K-1}\Pi_{j=0}^{k}\left(1-c_j\right)+\sum_{k=K}^{\infty}\Pi_{j=0}^{k}\left(1-c_j\right)=\infty,$$

matching our intuition, the chain is null recurrent.

## Question 9

### Part a

$$\begin{aligned} \mathbb{P}\left(X_2=0\mid X_1>0\right)&=\frac{\mathbb{P}\left(X_2=0,\,X_1>0\right)}{\mathbb{P}\left(X_1>0\right)}\\ &=\frac{\mathbb{P}\left(X_2=0\mid X_1=1\right)\mathbb{P}\left(X_1=1\right)+\mathbb{P}\left(X_2=0\mid X_1=3\right)\mathbb{P}\left(X_1=3\right)}{\mathbb{P}\left(X_1>0\right)}\\ &=\frac{0.5*0.1+(0.5)^3*0.4}{0.5}\\ &=0.2 \end{aligned}$$

### Part b

We have that

$$f(s) = \sum_{k=0}^{\infty} s^k p_k = 0.5 + 0.1s + 0.4s^3.$$

Hence,

$$\begin{aligned}
a_0 &= \mathbb{P}(X_0 = 0) = 0 && \text{(by assumption)} \\
a_1 &= \mathbb{P}(X_1 = 0) = f(a_0) = f(0) = 0.5, \\
a_2 &= \mathbb{P}(X_2 = 0) = f(a_1) = f(0.5) = 0.6, \\
a_3 &= \mathbb{P}(X_3 = 0) = f(a_2) = f(0.6) = 0.6464.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(X_3 = 0 \mid X_2 > 0) &= \frac{\mathbb{P}(X_3 = 0) - \mathbb{P}(X_2 = 0)}{\mathbb{P}(X_2 > 0)} \\
&= \frac{\mathbb{P}(X_3 = 0) - \mathbb{P}(X_2 = 0)}{1 - \mathbb{P}(X_2 = 0)} \\
&= \frac{0.6464 - 0.6}{1 - 0.6} \\
&= 0.116.
\end{aligned}$$

## Question 11

Inspired by the original branching process formulation, we need to ensure that the effective expected population growth for each individual is not bigger than one. Let $\mu$ be the effective expected population form one individual in one step, then:

$$\mu = \sum_{i=0}^{\infty} i \cdot p_i + 1 \cdot (1 - q).$$

We want $\mu \leq 1$, therefore

$$\mu = \sum_{i=0}^{\infty} i \cdot p_i + 1 \cdot (1 - q) \leq 1,$$

which gives the following that completes our answer:

$$\sum_{i=0}^{\infty} i \cdot p_i \leq q.$$

## Question 15

To solve this question we obtained assistance from Google Gemini and Claude AI models.

### Part a

For $x = 0$, the inequality is immediate. Also note that

$$G_n(x) = \mathbb{E}\left[\sum_{j=0}^{n} \mathbb{I}[S_j = x]\right] = \sum_{j=0}^{n} \mathbb{P}(S_j = x).$$

Let $x > 0$, and $T_x$ be the first time that $S_j$ is equal to $x$. Thus,

$$\mathbb{P}(S_j = x) = \sum_{k=0}^{\infty} \mathbb{P}(S_j = x, T_x = k).$$

So we have,

$$
\begin{aligned}
G_n(x) &= \mathbb{E}\left[\sum_{j=0}^{n} \mathbb{I}\left[S_j = x\right]\right] \\
&= \sum_{j=0}^{n} \mathbb{P}\left(S_j = x\right) \\
&= \sum_{j=0}^{n}\sum_{k=0}^{\infty} \mathbb{P}\left(S_j = x, T_x = k\right) \\
&= \sum_{j=0}^{n}\sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right)\mathbb{P}\left(S_j = x \mid T_x = k\right) \\
&= \sum_{k=0}^{\infty}\sum_{j=k}^{n} \mathbb{P}\left(T_x = k\right)\mathbb{P}\left(S_j = x \mid T_x = k\right) && \text{(Fubini)} \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right)\sum_{j=k}^{n}\mathbb{P}\left(S_j = x \mid T_x = k\right) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right)\sum_{j=k}^{n}\mathbb{P}\left(S_j = S_k \mid T_x = k\right) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right)\sum_{j=k}^{n}\mathbb{P}\left(S_{j-k} = 0\right) && (*) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right)\sum_{\ell=0}^{n-k}\mathbb{P}\left(S_\ell = 0\right) && \text{(probabilities are non-negative)} \\
&\le \sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right)\sum_{\ell=0}^{n}\mathbb{P}\left(S_\ell = 0\right) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(T_x = k\right) G_n(0) \\
&= \mathbb{P}\left(T_x \le n\right) G_n(0) \\
&\le G_n(0). && \text{(probabilities are in } [0,1])
\end{aligned}
$$

The application of Fubini's lemma is justified by reaching the upper bound $G_n(0)$ that is finite in the finite $n$ steps. The part marked by (*) is valid because the probability of revisiting state $k$, when starting in state $k$ is independent of the path taken up to time $k$, i.e., it's new random walk starting from 0.

### Part b

First note that

$$\sum_{x\in\mathbb{Z}} G_n(x) = \sum_{x\in\mathbb{Z}}\sum_{j=0}^{n} \mathbb{P}\left(S_j = x\right) \overset{\text{(Fubini)}}{=} \sum_{j=0}^{n}\sum_{x\in\mathbb{Z}} \mathbb{P}\left(S_j = x\right) = \sum_{j=0}^{n} 1 = n+1.$$

So,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{x\in\mathbb{Z}} G_n(x) = 1.$$

On the other hand,

$$\sum_{x\in\mathbb{Z}} G_n(x) = \sum_{|x|\le n\epsilon} G_n(x) + \sum_{|x|>n\epsilon} G_n(x).$$

So if we want to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{|x|\le n\epsilon}G_n(x)=1,$$

then we need to show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{|x|>n\epsilon}G_n(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{|x|>n\epsilon}\sum_{j=0}^{n}\mathbb{P}\left(S_j=x\right)\to 0.$$

First, we have the identity

$$\sum_{|x|>n\epsilon}\sum_{j=0}^{n}\mathbb{P}\left(S_j=x\right)=\sum_{j=0}^{n}\sum_{|x|>n\epsilon}\mathbb{P}\left(S_j=x\right)=\sum_{j=0}^{n}\mathbb{P}\left(|S_j|>n\epsilon\right).$$

So,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\mathbb{P}\left(|S_j|>n\epsilon\right)\to 0$$

is equivalent to saying that there exists an $N>0$ such that for $n\ge N$ and any $\delta>0$:

$$\left(\frac{1}{n}\sum_{j=0}^{n}\mathbb{P}\left(|S_j|>n\epsilon\right)\right)<\delta.$$

The above display is what we are going to prove.

Let us break down the summation into two parts for small positive real number $\alpha$ that its value will be specified later:

$$\sum_{j=0}^{n}\mathbb{P}\left(|S_j|>n\epsilon\right)=\underbrace{\sum_{j=0}^{\lfloor\alpha n\rfloor}\mathbb{P}\left(|S_j|>n\epsilon\right)}_{A}+\underbrace{\sum_{j=\lfloor\alpha n\rfloor+1}^{n}\mathbb{P}\left(|S_j|>n\epsilon\right)}_{B}.$$

For the first term $A$ we have:

$$A=\sum_{j=0}^{\lfloor\alpha n\rfloor}\mathbb{P}\left(|S_j|>n\epsilon\right)\le\sum_{j=0}^{\lfloor\alpha n\rfloor}1=\lfloor\alpha n\rfloor+1.$$

So,

$$\lim_{n\to\infty}\frac{1}{n}\cdot A=\alpha.$$

For the first term $B$ we have:

$$\lfloor\alpha n\rfloor<\lfloor\alpha n\rfloor+1\le j\le n\Rightarrow\frac{n}{j}\ge 1\text{ and }j\ge\alpha n.$$

Since $\frac{n}{j}\ge 1$:

$$\left\{\frac{S_j}{j}>\epsilon\frac{n}{j}\right\}\subseteq\left\{\frac{S_j}{j}>\epsilon\right\}\Rightarrow\mathbb{P}\left(\frac{S_j}{j}>\epsilon\frac{n}{j}\right)\le\mathbb{P}\left(\frac{S_j}{j}>\epsilon\right).$$

Since $j\ge\alpha n$, as n tends to infinity, $j$ tends to infinity, and the law of large numbers states that for large $n$ and some real number $\eta>0$:

$$\mathbb{P}\left(\frac{S_j}{j}>\epsilon\right)<\eta.$$

So,

$$B=\sum_{j=\lfloor\alpha n\rfloor+1}^{n}\mathbb{P}\left(|S_j|>n\epsilon\right)\le(n-\lfloor\alpha n\rfloor)\eta.$$

And,

$$\lim_{n\to\infty} \frac{1}{n} \cdot B \leq (1-\alpha)\eta < \eta.$$

By choosing $\alpha = \eta = \frac{\delta}{2}$and making $\delta$ tend to zero, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \mathbb{P}\left(|S_j| > n\epsilon\right) \to 0,$$

or equivalently there exists an $N > 0$ such that for $n \geq N$ and any $\delta > 0$:

$$\left( \frac{1}{n} \sum_{j=0}^{n} \mathbb{P}\left(|S_j| > n\epsilon\right) \right) < \delta.$$

**Part c**

Using Part (a) we have:

$$\sum_{|x|\leq n\epsilon} G_n(x) \leq \sum_{|x|\leq n\epsilon} G_n(0) = (2\lfloor n\epsilon \rfloor + 1)G_n(0).$$

We divide both sides by $n$ and take the limit as n tends to infinity. Then, we use Part (b):

$$1 \leq 2\epsilon G_n(0) \Rightarrow \frac{1}{2\epsilon} \leq G_n(0).$$

Choosing $\epsilon = \frac{1}{2M}, M \neq 0$ completes the proof. For $M = 0$ the inequality is immediate as shown in Part (a).

**Part d**

We need to prove that the Markov property holds on the chain and the chain is recurrent.

The chain is Markov because the value of $S_n$ is predicted using only the value of $S_{n-1}$. More explicitly, $S_n = S_{n-1} + X_n$, so the only information required from past is $S_{n-1}$.

To show that the chain is recurrent, we need to show that the expected number of visits to each state is infinite. Let $V_k$ denote the number of visits to state $k$, so $G_n(k) = \mathbb{E}[V_k]$. First consider state 0. $G_n(0) = G_{n-1}(0) + \mathbb{P}(S_n = 0) \geq G_{n-1}(0)$. So, $G_n(0)$ is an increasing sequence. We sowed in Part (c) that $G_n(0)$ is unbounded above. Consequently, for an increasing sequence that is unbounded above we have

$$\mathbb{E}[V_0] = \lim_{n\to\infty} G_n(0) = \infty.$$

To prove that for all other states $k \neq 0$, $\mathbb{E}[V_k] = \infty$. We show that the chain is irreducible and since the expected visits to state 0 is infinite, the expected visits to all other states are also infinite. To prove the irreducibility, note that by the assumption $\mathbb{E}[X_n] = 0$, $n \in \{0, 1, 2, \dots\}$. By independence of $X_n$s, we have that $\mathbb{E}[S_n] = 0$, $n \in \{0, 1, 2, \dots\}$. Since the random walk is moving on $\mathbb{Z}$, $\mathbb{E}[S_n] = 0$ means that the chain is not drifting toward $\pm\infty$ and there is a non-zero chance both states $k$ and $-k$ are reached, which proves that the chain irreducible.

**3.1** Suppose that the number of calls per hour arriving at an answering service follows a Poisson process with $\lambda = 4$.

(a) What is the probability that fewer than two calls come in the first hour?

(b) Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?

(c) The person answering the phones waits until fifteen phone calls have arrived before going to lunch. What is the expected amount of time that the person will wait?

(d) Suppose it is known that exactly eight calls arrived in the first two hours. What is the probability that exactly five of them arrived in the first hour?

(e) Suppose it is known that exactly $k$ calls arrived in the first four hours. What is the probability that exactly $j$ of them arrived in the first hour?

**3.3** Suppose $X_t$ and $Y_t$ are independent Poisson processes with parameters $\lambda_1$ and $\lambda_2$, respectively, measuring the number of calls arriving at two different phones. Let $Z_t = X_t + Y_t$.

(a) Show that $Z_t$ is a Poisson process. What is the rate parameter for $Z$?

(b) What is the probability that the first call comes on the first phone?

(c) Let $T$ denote the first time that at least one call has come from each of the two phones. Find the density and distribution function of the random variable $T$.

**3.5** Let $X_t$ be a Markov chain with state space $\{1, 2\}$ and rates $\alpha(1, 2) = 1, \alpha(2, 1) = 4$. Find $\mathbf{P}_t$.

**3.10** Suppose $\alpha$ gives the rates for an irreducible continuous-time Markov chain on a finite state space. Suppose the invariant probability measure is $\pi$. Let

$$p(x, y) = \alpha(x, y)/\alpha(x), \quad x \neq y,$$

be the transition probability for the discrete-time Markov chain corresponding to the continuous-time chain "when it moves." Find the invariant probability for the discrete-time chain in terms of $\pi$ and $\alpha$.

**3.11** Let $X_t$ be a continuous-time birth-and-death process with birth rate $\lambda_n = 1 + (1/(n+1))$ and death rate $\mu_n = 1$. Is this process positive recurrent, null recurrent, or transient? What if $\lambda_n = 1 - (1/(n+2))$?

**3.14** Consider a birth-and-death process with $\lambda_n = 1/(n+1)$ and $\mu_n = 1$. Show that the process is positive recurrent and give the stationary distribution.

# Answers to Week Six's Assignments

Alireza Kazemipour

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## Continuous-Time Markov Chains

### Question 1

Note that in a Poisson process $\mathbb{P}(X_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$.

#### Part a

$$\mathbb{P}(X_1 < 2) = \mathbb{P}(X_1 = 0) + \mathbb{P}(X_1 = 1) = e^{-4} + 4e^{-4} = 5e^{-4}.$$

#### Part b

Because the arrivals in different intervals are independent, the information about the six calls in the first interval does not effect the probability of having at least two calls in the second interval. Similar to Part (a), for any single interval (e.g., the second interval) such that at least two calls occur we have:

$$\mathbb{P}(X_2 \geq 2) = 1 - \mathbb{P}(X_2 < 2) = 1 - \mathbb{P}(X_1 < 2) = 1 - 5e^{-4}.$$

#### Part c

Let $Y_{15}$ denote the amount of time the person has to wait. We know that each arrival time has an exponential distribution with mean $\lambda$, so $\mathbb{E}[Y_{15}] = \frac{15}{\lambda} = \frac{15}{4} = 3.75$ hours.

#### Part d

Due to independence of calls' arrival in different intervals, the probability that a call has happened between *two consecutive* intervals is $\frac{1}{2}$. Using the binomial distribution we have:

$$\mathbb{P}(X_1 = 5 \mid X_1 + X_2 = 8) = \binom{8}{5} \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{8-5} = \binom{8}{5} \cdot \left(\frac{1}{2}\right)^8.$$

#### Part e

This part is the generalization of Part (d).

$$\mathbb{P}(X_1 = j \mid X_1 + X_2 + X_3 + X_4 = k) = \binom{k}{j} \cdot \left(\frac{1}{4}\right)^j \cdot \left(\frac{3}{4}\right)^{k-j}.$$

### Question 3

#### Part a

$$\begin{aligned}
\mathbb{P}(Z_t = k) &= \mathbb{P}(X_t + Y_t = k) \\
&= \sum_{i=0}^{k} \mathbb{P}(X_t = i, Y_t = k - i) \\
&= \sum_{i=0}^{k} \mathbb{P}(X_t = i)\,\mathbb{P}(Y_t = k - i) && \text{(independence)} \\
&= \sum_{i=0}^{k} e^{-\lambda_1 t} \frac{(\lambda_1 t)^i}{i!} \cdot e^{-\lambda_2 t} \frac{(\lambda_2 t)^{k-i}}{(k-i)!} \\
&= e^{-(\lambda_1+\lambda_2)t} t^k \sum_{i=0}^{k} \frac{(\lambda_1)^i}{i!} \cdot \frac{(\lambda_2)^{k-i}}{(k-i)!} \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{t^k}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \cdot \lambda_1^i \lambda_2^{k-i} \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{t^k}{k!} \sum_{i=0}^{k} \binom{k}{i} \cdot \lambda_1^i \lambda_2^{k-i} \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{t^k}{k!} (\lambda_1 + \lambda_2)^k && \text{(Binomial theorem)} \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{(t(\lambda_1 + \lambda_2))^k}{k!},
\end{aligned}$$

where the last line is the Poisson distribution with parameter $\lambda_1 + \lambda_2$. Hence, $Z_t$ follows a Poisson distribution with the rate parameter $\lambda_1 + \lambda_2$.

#### Part b

Let $T_X, T_Y$ be the first times that the first calls arrive on the first and the second phones respectively. $T_X, T_Y$ are independent and follow the exponential distributions with parameters $\lambda_1, \lambda_2$. We have:

$$\begin{aligned}
\mathbb{P}(T_X < T_Y) &= \int_0^\infty \mathbb{P}(T_Y > x)\, dF_{T_X}(x) \\
&= \int_0^\infty \left( \int_x^\infty dF_{T_Y}(y) \right) f_{T_X}(x) dx \\
&= \int_0^\infty \left( \int_x^\infty \lambda_2 e^{-\lambda_2 y} dy \right) f_{T_X}(x) dx \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} dx \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

#### Part c

Let $T_X, T_Y$ denote the first time that one call has come from each of the phones respectively. $T_X, T_Y$ follow the exponential distributions with parameters $\lambda_1, \lambda_2$. Also, $T = \max(T_X, T_Y)$, so we have:

$$\begin{aligned}
\mathbb{P}(T \le t) &= \mathbb{P}(T_X \le t, T_Y \le t) \\
&= \mathbb{P}(T_X \le t)\,\mathbb{P}(T_Y \le t) && \text{(independence)} \\
&= \left(1 - e^{-\lambda_1 t}\right)\left(1 - e^{-\lambda_2 t}\right) \\
&= F_T(t).
\end{aligned}$$

To find the probability density function, we take the derivative of $F_T(t)$ with respect to $t$:

$$\frac{d}{dt}F_T(t) = \lambda_1 e^{-\lambda_1 t}\left(1 - e^{-\lambda_2 t}\right) + \lambda_2 e^{-\lambda_2 t}\left(1 - e^{-\lambda_1 t}\right) = \lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}.$$

## Question 5

$$A = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}, \quad \text{and} \quad \det(A - \lambda I) = 0 \Rightarrow \lambda = 0, -5.$$

So,

$$A = QDQ^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}.$$

And,

$$\begin{aligned} P_t = Q\left(\sum_{n=0}^{\infty}(tD)^n\right)Q^{-1} &= \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-5t} \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 4 + e^{-5t} & 1 - e^{-5t} \\ 4 - 4e^{-5t} & 1 + 4e^{-5t} \end{pmatrix}. \end{aligned}$$

## Question 10

Let $A$ be the infinitesimal generator matrix. In the continuous-time chain we have:

$$\pi A = 0 \Rightarrow -\alpha(x)\pi(x) + \sum_{y \neq x}\pi(y)\alpha(y,x) = 0 \Rightarrow \pi(x) = \sum_{y\neq x}\pi(y)\frac{\alpha(y,x)}{\alpha(x)}.$$

Let $\mu$ be the invariant probability distribution in the discrete-time chain. We have:

$$\mu = \mu P \xrightarrow{(p(x,x)=0)} \mu(x) = \sum_{y\neq x}\mu(y)p(y,x) \Rightarrow \mu(x) = \sum_{y\neq x}\mu(y)\frac{\alpha(y,x)}{\alpha(y)}.$$

Since

$$\begin{aligned} \pi(x) &= \sum_{y\neq x}\pi(y)\frac{\alpha(y,x)}{\alpha(x)}, \\ \mu(x) &= \sum_{y\neq x}\mu(y)\frac{\alpha(y,x)}{\alpha(y)}, \text{ and} \\ \sum_x \pi(x) &= \sum_x \mu(x) = 1, \end{aligned}$$

we conjecture that $\mu(x) = C\cdot\pi(x)\cdot\alpha(x)$. Plugging this guess for $\mu$ into $\mu(x) = \sum_{y\neq x}\mu(y)\frac{\alpha(y,x)}{\alpha(y)}$ we get:

$$C\cdot\pi(x)\cdot\alpha(x) = \sum_{y\neq x}C\pi(y)\alpha(y)\frac{\alpha(y,x)}{\alpha(y)} = C\sum_{y\neq x}\pi(y)\alpha(y,x).$$

By $\pi(x) = \sum_{y\neq x}\pi(y)\frac{\alpha(y,x)}{\alpha(x)}$:

$$C\sum_{y\neq x}\pi(y)\alpha(y,x) = C\cdot\pi(x)\cdot\alpha(x),$$

confirming that our guess was correct. To find $C$ we use $\sum_y \mu(y) = 1$:

$$\sum_y \mu(y) = 1 \Rightarrow \sum_y C\pi(y)\alpha(y) = 1 \Rightarrow C = \frac{1}{\sum_y \pi(y)\alpha(y)}.$$

Thus,

$$\mu(x) = \frac{\pi(x)\alpha(x)}{\sum_y \pi(y)\alpha(y)}.$$

**Question 11**

- $\lambda_n = 1 + \frac{1}{n+1} = \frac{n+2}{n+1}$, $\mu_n = 1$:

$$\sum_{k=1}^{\infty} \Pi_{n=1}^{k} \frac{\mu_n}{\lambda_n} = \sum_{k=1}^{\infty} \Pi_{n=1}^{k} \frac{n+1}{n+2} = \sum_{k=1}^{\infty} \Pi_{l=2}^{k+1} \frac{l}{l+1} = \sum_{k=1}^{\infty} \frac{2}{k+2} \overset{(\sum_{i=1}^{\infty} \frac{1}{i} = \infty)}{=} \infty \Rightarrow \text{ the chain is recurrent.}$$

$$1 + \sum_{k=1}^{\infty} \Pi_{n=0}^{k-1} \frac{\lambda_n}{\mu_{n+1}} = 1 + \sum_{k=1}^{\infty} \Pi_{n=0}^{k-1} \frac{n+2}{n+1} = 1 + \sum_{k=1}^{\infty} \Pi_{l=1}^{k} \frac{l+1}{l} = 1 + \sum_{k=1}^{\infty} \frac{k+1}{1} = \infty \Rightarrow \text{ the chain is null recurrent.}$$

- $\lambda_n = 1 - \frac{1}{n+2} = \frac{n+1}{n+2}$, $\mu_n = 1$:

$$\sum_{k=1}^{\infty} \Pi_{n=1}^{k} \frac{\mu_n}{\lambda_n} = \sum_{k=1}^{\infty} \Pi_{n=1}^{k} \frac{n+2}{n+1} = \sum_{k=1}^{\infty} \Pi_{l=2}^{k+1} \frac{l+1}{l} = \sum_{k=1}^{\infty} \frac{k+2}{2} = \infty \Rightarrow \text{ the chain is recurrent.}$$

$$1 + \sum_{k=1}^{\infty} \Pi_{n=0}^{k-1} \frac{\lambda_n}{\mu_{n+1}} = 1 + \sum_{k=1}^{\infty} \Pi_{n=0}^{k-1} \frac{n+1}{n+2} = 1 + \sum_{k=1}^{\infty} \Pi_{l=1}^{k} \frac{l}{l+1} = 1 + \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty \Rightarrow \text{ the chain is null recurrent.}$$

**Question 14**

$$q = 1 + \sum_{k=1}^{\infty} \Pi_{n=0}^{k-1} \frac{\lambda_n}{\mu_{n+1}} = 1 + \sum_{k=1}^{\infty} \Pi_{n=0}^{k-1} \frac{1}{n+1} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} = e < \infty \Rightarrow \text{ positive recurrence.}$$

$$\pi(k) = \frac{\lambda_0 \dots \lambda_{k-1}}{\mu \dots \mu_k} q^{-1} = \frac{1}{ek!}.$$

**4.2** The following game is played: you roll two dice. If you roll a 7, the game is over and you win nothing. Otherwise, you may stop and receive an amount equal to the sum of the two dice. If you continue, you roll again. The game ends whenever you roll a 7 or whenever you say stop. If you say stop before rolling a 7 you receive an amount equal to the sum of the two dice on the last roll. What is your expected winnings: a) if you always stop after the first roll; b) if you play to optimize your expected winnings?

**4.4** Consider Exercise 4.2. Do the problem again assuming:
(a) a cost function of $g = [2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1]$;
(b) a discount factor $\alpha = .8$;
(c) both.

**4.7** If $u_1(y), u_2(y), \ldots$ are all functions that are superharmonic at $x$ for $\mathbf{P}$, i.e.,

$$\mathbf{P}u_i(x) \leq u_i(x),$$

and we let $u$ be the function

$$u(y) = \inf_i u_i(y),$$

show that $u$ is superharmonic at $x$ for $\mathbf{P}$.

**4.9** Suppose $X_n$ is random walk with absorbing boundary on $\{0, 1, 2, \ldots\}$ with

$$p(n, n+1) = p(n, n-1) = \frac{1}{2}, \quad n \geq 1.$$

Suppose our payoff function is $f(n) = n^2$. Let us try to find a stopping time $T$ that will maximize $\mathbb{E}[f(X_T)]$.

(a) Show that if $X_n > 0$, then

$$\mathbb{E}[f(X_{n+1}) \mid X_n] > f(X_n).$$

Conclude that any optimal strategy does not stop at any integer greater than 0.

(b) Since the random walk is recurrent, we know that we will eventually reach 0 at which point we stop and receive a payoff of 0. Since our "optimal" strategy tells us never to stop before then, our eventual payoff in the optimal strategy is 0. Clearly something is wrong here—any ideas?

# Answers to Homework 7

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## Optimal Stopping

### Question 2

#### Part a

There are 36 possible combinations in total for number between 2 and 12. We list the ways that each individual outcome can appear:

| state ($i$) | count $n(i)$ | combinations |
|---|---|---|
| 2 | 1 | $1+1$ |
| 3 | 2 | $1+2$, $2+1$ |
| 4 | 3 | $1+3$, $3+1$, $2+2$ |
| 5 | 4 | $1+4$, $4+1$, $2+3$, $3+2$ |
| 6 | 5 | $1+5$, $5+1$, $4+2$, $2+4$, $3+3$ |
| 7 (Game over) | 6 | $3+4$, $4+3$, $1+6$, $6+1$, $2+5$, $5+2$ |
| 8 | 5 | $4+4$, $5+3$, $3+5$, $6+2$, $2+6$ |
| 9 | 4 | $6+3$, $3+6$, $4+5$, $5+4$ |
| 10 | 3 | $5+5$, $6+4$, $4+6$ |
| 11 | 2 | $6+5$, $5+6$ |
| 12 | 1 | $6+6$ |

The expected winning after the first roll is equal to

$$\frac{1}{36}\sum_{i=2}^{12} i * n(i) - 7 * \frac{6}{36} = \frac{42-7}{6} \approx 5.83.$$

#### Part b

Since the expected winning after the first roll is 5.83, the optimal strategy for each of $1, 2, \ldots, 5$ is to continue and for each of $8, 9, \ldots, 12$ is to stop. Since 7 is terminating, the only state that we need to compute the optimal strategy for is state 6. Suppose the optimal strategy in this state was to continue, then by defining $u$ as the optimal expected payoff for this state we have:

$$u = \mathbb{P}(i \leq 6)\, u + \frac{1}{36}\sum_{i=8}^{12} i * n(i) \Rightarrow \frac{21}{36}u = \frac{140}{36} \Rightarrow u \approx 6.66 > 6 = f(6).$$

So, the optimal strategy in state 6 is also to continue. Hence, the game only stops at states $7, 8, 9, 10, 11, 12$ and the expected winning is:

$$6.66 * \mathbb{P}(i \leq 6) + \frac{1}{36}\sum_{i=8}^{12} i * n(i) = \frac{15}{36} \cdot \frac{140}{21} + \frac{140}{36} = \frac{140}{21} = 6.66.$$

## Question 4

### Part a

We pay $g(i)$ at state $i$. Define the effective one step payoff $h = f - g$ as follows

| state $(j) = h(j)$ | count $n(j)$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 3 |
| 3 | 4 |
| 5 | 5 |
| 6 (Game over) | 6 |
| 7 | 5 |
| 8 | 4 |
| 9 | 3 |
| 10 | 2 |
| 11 | 1 |

- We stop on the first roll:

$$\frac{1}{36}\sum_{j=0,j\neq 6}^{11} h(j) * n(j) =$$
$$\frac{0*1+1*2+2*3+3*4+5*5+7*5+8*4+9*3+10*2+11*1}{36} = \frac{170}{36} \approx 4.72.$$

- We play to optimize:

  Since the expected effective payoff at the first roll is 4.72, for states bigger or equal to 8 we should stop. At the same time, since the expected effective payoff at the first roll is 4.72, it is logical to continue at states $i = 2, 3, 4, 5$ because if we stop at state 5, then $5 - 2 < 4.72$. The only non-obvious state to continue or stop is state 6. Let us assume that we continue playing in state 6 as well. The expected effective payoff $u$ would be:

$$u = \mathbb{P}(i \leq 6)\, u + \frac{1}{36}\sum_{j=7}^{11} h(j) * n(j) \Rightarrow \left(1 - \frac{15}{36}\right) u = \frac{125}{36} \Rightarrow u = \frac{125}{21} > 5 = f(6) - 1.$$

  So, we should continue at state 6 as well. Consequently the expected effective winning is equal to:

$$\mathbb{P}(i \leq 6) * \frac{125}{21} + \frac{1}{36}\sum_{j=7}^{11} h(j) * n(j) = \frac{125}{21}$$

### Part b

We discount future returns by $\alpha$.

- We stop on the first roll:

  The discounting would be irrelevant and the same as Question 2 Part (a), we get:

$$\frac{1}{36}\sum_{i=2}^{12} i * n(i) - 7 * \frac{6}{36} = \frac{42-7}{6} \approx 5.83.$$

- We play to optimize:

  Since the discounted expected payoff after the first roll is $0.8 * 5.83 = 4.66$, for states bigger or equal to 5 we should stop. At the same time, Since the discounted expected payoff after the first roll is 4.66, it is logical to continue at states $i = 2, 3, 4$ because if we stop at state 4, then $4 < 4.72$.

We need to make sure the discounted expected payoff of continuing at state 4 is bigger than 4. The discounted expected payoff $u$ would be:

$$u = \alpha\,\mathbb{P}\,(i \leq 4)\,u + \frac{1}{36}\sum_{i=5, i\neq 7}^{12} i * n(i) \Rightarrow$$
$$(1 - 0.8 * \frac{1}{6})u = \frac{190}{36} \Rightarrow u = \frac{475}{78} \approx 6 > 4 = f(4).$$

So, we correctly continue at state 4. Consequently, the discounted expected winning is equal to:

$$0.8 * \mathbb{P}\,(i \leq 4) * \frac{475}{78} + \frac{1}{36}\sum_{i=5, i\neq 7}^{12} i * n(i) = \frac{475}{78} \approx 6.$$

### Part c

We discount future returns by $\alpha$.

- We stop on the first roll:

  The discounting would be irrelevant and the same as Part (a) we get:

  $$\frac{1}{36}\sum_{j=0, j\neq 6}^{11} h(j) * n(j) =$$
  $$\frac{0*1+1*2+2*3+3*4+5*5+7*5+8*4+9*3+10*2+11*1}{36} = \frac{170}{36} \approx 4.72.$$

- We play to optimize:

  Since the discounted effective expected payoff after the first roll is $0.8 * 4.72 = 3.77$, for states bigger or equal to 6 we should stop. At the same time, Since the discounted effective expected payoff after the first roll is 3.77, it is logical to continue at states $i = 2, 3, 4, 5$ because if we stop at state 5, then $5 - 2 < 3.77$. The discounted effective expected payoff $u$ at state 5 would be:

  $$u = \alpha\,\mathbb{P}\,(i \leq 5)\,u + \frac{1}{36}\sum_{j=5, j\neq 6}^{11} h(j) * n(j) \Rightarrow$$
  $$(1 - 0.8 * \frac{10}{36})u = \frac{150}{36} \Rightarrow u = \frac{150}{28} \approx 5.35 > 5 = f(5).$$

  So, we correctly continue at state 5. Consequently, the discounted expected winning is equal to:

  $$0.8 * \mathbb{P}\,(i \leq 5) * 5.35 + \frac{1}{36}\sum_{j=5, j\neq 6}^{11} h(j) * n(j) \approx 5.35.$$

## Question 7

Let $u(x) = \inf_i u_i(x)$. Fix an arbitrary index $i$. Since $P$ is positive linear operator we have,

$$Pu(x) \leq Pu_i(x) \leq u_i(x).$$

Since $i$ was chosen arbitrary we have

$$Pu(x) \leq \inf_i u_i(x) = u(x),$$

which completes the proof.

**Question 9**

**Part a**

First we show when $X_n > 0$ it is always the case that

$$\mathbb{E}\left[f(X_{n+1}) \mid X_n\right] > f(X_n).$$

Let $X_n$ be any positive integer $k$. Then, $f(X_n) = k^2$. On the other hand,

$$\mathbb{E}\left[f(X_{n+1}) \mid X_n = k\right] = \frac{1}{2}(k-1) + \frac{1}{2}(k+1) = k^2 + 1 > k = f(X_n),$$

which completes the first part of the argument. Now, to show no optimal strategy stops at any positive integer, i.e., $T = \infty$, we use the tower rule of expectations. Let $T = t$, where $t$ is any positive integer. We have

$$\mathbb{E}\left[f(X_T)\right] = \mathbb{E}\left[f(X_t)\right] = \mathbb{E}\left[\mathbb{E}\left[f(X_t) \mid X_{t-1}\right]\right] \overset{\text{(proved above)}}{>} \mathbb{E}\left[f(X_{t-1})\right].$$

So we proved that for any positive integer $t$ such that $T = t$, we have $\mathbb{E}\left[f(X_t)\right] > \mathbb{E}\left[f(X_{t-1})\right]$. Therefore, to let $\mathbb{E}\left[f(X_t)\right]$ increase, we should let $t$ tend to infinity, which completes the proof.

**Part b**

The flaw is happening in interpreting the max and sup operators. The optimal strategy chooses a $T$ that *maximizes* $\mathbb{E}\left[f(X_T)\right]$. But, $\mathbb{E}\left[f(X_T)\right]$ has the values in $[0, \infty)$ and is monotonically increasing in $T$. So, while $\sup\{\mathbb{E}\left[f(X_T)\right]\} = \infty$, its maximum does not exists. Therefore, despite reasoning about it in the previous part, *no optimal strategy* exists that can name a positive integer $T$ maximizing $\mathbb{E}\left[f(X_T)\right]$.

**5.2** Suppose that $X_t$ is a Poisson process with parameter $\lambda = 1$. Find $E(X_1 \mid X_2)$ and $E(X_2 \mid X_1)$.

**5.4** Let $X_1, X_2, X_3, \ldots$ be independent identically distributed random variables. Let $m(t) = \mathbb{E}\left(e^{tX_1}\right)$ be the moment generating function of $X_1$ (and hence of each $X_i$). Fix $t$ and assume $m(t) < \infty$. Let $S_0 = 0$ and for $n > 0$,

$$S_n = X_1 + \cdots + X_n.$$

Let $M_n = m(t)^{-n} e^{tS_n}$. Show that $M_n$ is a martingale with respect to $X_1, X_2, \ldots$.

# Answers to Homework 8

Alireza Kazemipour

December 16, 2025

## Martingales

### Question 2

$X_2$ is the number of customers by the end of the second interval. We write $X_2$ as:

$$X_2 = X_1 + (X_2 - X_1).$$

The number of arrivals during the second interval, i.e., $(X_2 - X_1)$ is independent of arrivals in the first interval, i.e., $X_1$, hence $\mathbb{E}[X_2 - X_1 \mid X_1] = \mathbb{E}[X_2 - X_1] = \lambda = 1$. Therefore, we have

$$\mathbb{E}[X_2 \mid X_1] = X_1 + \mathbb{E}[X_2 - X_1 \mid X_1] = X_1 + 1.$$

For the second part, suppose $X_2 = n$. If during the first two intervals $n$ customers have arrived, due to independence of arrivals in each interval, each customer could have arrived with 50% probability during each of the intervals. So,

$$\mathbb{E}[X_1 \mid X_2 = n] = \frac{n}{2} = \frac{X_2}{2}.$$

### Question 4

Let $\mathcal{F}_n$ denote the filtration until and including step $n$. We need to show

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n.$$

We have,

$$\begin{aligned}
\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\left[m(t)^{-(n+1)} e^{tS_{n+1}} \mid \mathcal{F}_n\right] \\
&= \mathbb{E}\left[m(t)^{-(n+1)} e^{tS_n} \cdot e^{tX_{n+1}} \mid \mathcal{F}_n\right] \\
&= m(t)^{-(n+1)} e^{tS_n}\, \mathbb{E}\left[e^{tX_{n+1}} \mid \mathcal{F}_n\right]. && (S_n \text{ is } \mathcal{F}_n\text{-measurable})
\end{aligned}$$

Since random variables are independently identically distributed, given $\mathcal{F}_n$, $X_{n+1}$ acts the same as $X_1$ given $\mathcal{F}_0$. Hence, $\mathbb{E}\left[e^{tX_{n+1}} \mid \mathcal{F}_n\right] = \mathbb{E}\left[e^{tX_1}\right] = m(t)$. Therefore, the proof is completed as

$$\begin{aligned}
\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= m(t)^{-(n+1)} e^{tS_n}\, \mathbb{E}\left[e^{tX_{n+1}} \mid \mathcal{F}_n\right] \\
&= m(t)^{-(n+1)} e^{tS_n} \cdot m(t) \\
&= m(t)^{-n} e^{tS_n} \\
&= M_n.
\end{aligned}$$

**8.1** Let $X$ be a normal random variable, mean 0 variance 1. Show that if $a > 0$

$$\mathbb{P}\{X \geq a\} \leq \frac{2}{a\sqrt{2\pi}} e^{-a^2/2}.$$

(Hint:

$$\int_a^\infty e^{-x^2/2}\, dx \leq \int_a^\infty e^{-ax/2}\, dx.)$$

**8.3** Let $X_{n1}, \dots, X_{nn}$ be independent Poisson random variables with mean $1/n$. Then

$$X = X_{n1} + \cdots + X_{nn},$$

is a Poisson random variable with mean 1. Let

$$M_n = \max\{X_{n1}, \dots, X_{nn}\}.$$

Find

$$\lim_{n \to \infty} \mathbb{P}\{M_n > 1/2\}.$$

**8.6** If $Y_1, \dots, Y_n$ have a joint normal distribution with mean 0, then the covariance matrix is the matrix $\Gamma$ whose $(i, j)$ entry is $\mathbb{E}(Y_i Y_j)$. Let $X_t$ and $s_1, \dots, s_n$ be as in Exercise 8.5.

(a) Find the covariance matrix $\Gamma$ for $X_{s_1}, \dots, X_{s_n}$.

(b) The moment generating function (mgf) for $Y_1, \dots, Y_n$ is the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(t_1, \dots, t_n) = \mathbb{E}\left[e^{t_1 Y_1 + \cdots + t_n Y_n}\right].$$

Find the mgf for $Y_1, \dots, Y_n$ in terms of its covariance matrix $\Gamma$.

(c) If two distributions have the same mgf, then the two distributions are the same. Use this fact to prove the following: if $Y_1, \dots, Y_n$ have a mean 0 joint normal distribution, and $\mathbb{E}[Y_i Y_j] = 0$ for all $i \neq j$, then $Y_1, \dots, Y_n$ are independent.

**8.7** Suppose $X_t$ is a standard Brownian motion and $Y_t = a^{-1/2} X_{at}$ with $a > 0$. Show that $Y_t$ is a standard Brownian motion.

**8.8** Suppose $X_t$ is a standard Brownian motion and $Y_t = t X_{1/t}$. Show that $Y_t$ is a standard Brownian motion. (Hint: it may be useful to use Exercise 8.6 (c).)

# Answers to Homework 9

Alireza Kazemipour

December 16, 2025

## Brownian Motion

### Question 1

$$\mathbb{P}(X \geq a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{x^2}{2}} \overset{\text{(hint)}}{\leq} \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{ax}{2}} = \frac{2}{a\sqrt{2\pi}} e^{-\frac{a^2}{2}}.$$

We only need to prove the hint, given in the question, holds. Since the exponential function is non-negative, it suffices to show that $-x^2 \leq -ax$ for $x \geq a > 0$. Consider the function:

$$h(x) = -x^2 + ax, \quad x \geq a > 0.$$

We will show that $h \leq 0$ on its domain. The roots of $h$ are $a$ and $0$. According to the domain, the only acceptable root is $a$. Since $h$ is concave for any $x \geq a$, $h(x) \leq 0$ completing the proof.

### Question 3

Since $X_{ni},\ i = 1, 2, \ldots n$ are Poisson random variables, they only take values on non-negative integers. Hence, we have the following equalities between events:

$$\left\{M_n > \frac{1}{2}\right\} = \{M_n \geq 1\} = \{X_{n1} = 0, X_{n2} = 0, \ldots, X_{nn} = 0\}^{\complement}.$$

On the other hand we have:

$$\mathbb{P}(X_{n1} = 0, X_{n2} = 0, \ldots, X_{nn} = 0) = \Pi_{i=1}^n \mathbb{P}(X_{ni} = 0) = \Pi_{i=1}^n \left(e^{-1/n} \frac{i^0}{0!}\right) = \left(e^{-1/n}\right)^n = e^{-1}.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}(M_n > 1/2) = \lim_{n\to\infty} (1 - e^{-1}) = 1 - e^{-1}.$$

### Question 6

We obtained assistance in solving this question from Google Gemini model.

#### Part a

Let $i$ and $j$ be two positive integers that $i < j$. For a standard Brownian motion we have:

$$\mathbb{E}[X_i X_j] \overset{\text{(tower rule)}}{=} \mathbb{E}[\mathbb{E}[X_i X_j \mid \mathcal{F}_i]] = \mathbb{E}[X_i \mathbb{E}[X_j \mid \mathcal{F}_i]] \overset{\text{(Markov property)}}{=}$$
$$\mathbb{E}[X_i \cdot X_i] = \mathsf{Var}(X_i - X_0) = i = \min(i, j).$$

Using $s_1 \leq s_2 \leq \cdots \leq s_n$ notation instead of $i$ and $j$, we have:

$$\Gamma = \begin{pmatrix} s_1 & s_1 & \ldots & s_1 \\ s_2 & s_2 & \ldots & s_2 \\ \vdots & \vdots & \ldots & \vdots \\ s_n & s_n & \ldots & s_n \end{pmatrix}.$$

**Part b**

Let

$$Z = t_1 Y_1 + \cdots t_n Y_n.$$

Since $Z$ is the linear combination of normal random variables, it is also a normal random variable. Now we compute the mean and variance of $Z$. By the linearity of expectations, $\mathbb{E}[Z] = 0$. The variance of $Z$ is computed as follows:

$$\begin{aligned}
\sigma_z^2 = \mathsf{Var}(Z) = \mathsf{Cov}\left(\sum_{i=1}^n t_i Y_i, \sum_{j=1}^n t_j Y_j\right) \\
&= \mathbb{E}\left[\sum_{i=1}^n t_i Y_i \cdot \sum_{j=1}^n t_j Y_j\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^n t_i t_j Y_i Y_j\right] \\
&= \sum_{i=1}^n \sum_{j=1}^n t_i t_j \, \mathbb{E}[Y_i Y_j] \\
&= \sum_{i=1}^n \sum_{j=1}^n t_i t_j \Gamma_{ij} \\
&= \mathbf{t}^\top \Gamma \mathbf{t}.
\end{aligned}$$

We want to find $\mathbb{E}\left[e^Z\right]$, where $Z \sim \mathcal{N}(0, \sigma_z^2)$. Using the law of the unconscious statistician, we have:

$$\begin{aligned}
\mathbb{E}\left[e^Z\right] &= \frac{1}{\sqrt{2\pi\sigma_z^2}} \int_{-\infty}^{\infty} e^Z \cdot e^{-\frac{Z^2}{2\sigma_z^2}} dZ \\
&= \frac{1}{\sqrt{2\pi\sigma_z^2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2 + 2Z\sigma_z^2}{2\sigma_z^2}} dZ \\
&= \frac{1}{\sqrt{2\pi\sigma_z^2}} \int_{-\infty}^{\infty} e^{-\frac{Z^2 + 2Z\sigma_z^2 + \sigma_z^4 - \sigma_z^4}{2\sigma_z^2}} dZ \\
&= \frac{1}{\sqrt{2\pi\sigma_z^2}} \int_{-\infty}^{\infty} e^{-\frac{(Z + \sigma_z^2)^2 - \sigma_z^4}{2\sigma_z^2}} dZ \\
&= e^{\frac{\sigma_z^2}{2}} \cdot \underbrace{\frac{1}{\sqrt{2\pi\sigma_z^2}} \int_{-\infty}^{\infty} e^{-\frac{(Z + \sigma_z^2)^2}{2\sigma_z^2}} dZ}_{1} \qquad \text{(Shifting the mean preserves the probabilities' sum)} \\
&= \exp\left(\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n t_i t_j \Gamma_{ij}\right).
\end{aligned}$$

**Part c**

If $\mathbb{E}[Y_i Y_j] = 0$ for all $i \neq j$, then using Part b:

$$f(t_1, \cdots, t_n) = \exp\left(\frac{1}{2} \sum_{i=1}^n t_i^2 \mathsf{Var}(Y_i)\right) = \Pi_{i=1}^n \exp\left(\frac{1}{2} t_i^2 \mathsf{Var}(Y_i)\right).$$

But similar to our proof in Part b, $\exp\left(\frac{1}{2} t_i^2 \mathsf{Var}(Y_i)\right)$ is the MGF of $Y_i$ that has mean zero. So,

$$f(t_1, \cdots, t_n) = \Pi_{i=1}^n \exp\left(\frac{1}{2} t_i^2 \mathsf{Var}(Y_i)\right) = f_{Y_1}(t_1) \cdot f_{Y_2}(t_2) \cdot \ldots f_{Y_n}(t_n).$$

We also know that two random variables are independent if and only if their joint MGF is equal to the product of their marginal MGFs. And, since in the question it is hinted that two distributions are equal if and only if their MGFs are equal, we have proved MGF of $Y_1, Y_2, \cdots Y_n$ is equal to the MGF of independent random variables, hence $Y_1, Y_2, \cdots Y_n$ must be independent.

### Question 7

We need to prove the following four properties:

1. $Y_0 = 0$.

$$Y_0 = \frac{1}{\sqrt{a}} X_0 \overset{(X_0=0)}{=} 0.$$

2. For any $s_1 \le t_1 \le s_2 \le t_2 \cdots \le s_n \le t_n$, the random variables $Y_{t_1} - Y_{s_1}, \cdots Y_{t_n} - Y_{s_n}$ are independent.

   For any $i \neq j$ we have

$$\begin{aligned}\mathbb{E}\left[(Y_{t_i} - Y_{s_i})(Y_{t_j} - Y_{s_j})\right] &= \mathbb{E}\left[\frac{1}{\sqrt{a}}(X_{at_i} - X_{as_i})\frac{1}{\sqrt{a}}(X_{at_j} - X_{as_j})\right] \\ &\overset{(X_t \text{ is a Brownian motion})}{=} \mathbb{E}\left[\frac{1}{\sqrt{a}}(X_{at_i} - X_{as_i})\right]\mathbb{E}\left[\frac{1}{\sqrt{a}}(X_{at_j} - X_{as_j})\right] \\ &= \mathbb{E}\left[(Y_{t_i} - Y_{s_i})\right]\mathbb{E}\left[(Y_{t_j} - Y_{s_j})\right].\end{aligned}$$

3. For any $s < t$, the random variables $Y_t - Y_s$ has a normal distribution with mean zero and variance $(t-s)$.

   $Y_t = \frac{1}{\sqrt{a}} X_{at}$ is a multiple of a normal random variable, so it is also a normal random variable. Consequently, any linear combinations of $Y_t$ and $Y_s$ for $s < t$ is normal. Now we compute the mean and variance:

$$\mathbb{E}[Y_t - Y_s] = \frac{1}{\sqrt{a}}\mathbb{E}[X_{at} - X_{as}] = 0.$$
$$\mathsf{Var}(Y_t - Y_s) = \frac{1}{a}\mathsf{Var}(X_{at} - X_{as}) = \frac{a(t-s)}{a} = t - s.$$

4. The paths are continuous, i.e., the function $t \mapsto Y_t$ is a continuous function of $t$.

   Since $Y_t$ is a multiple of $X_{at}$ and $X_t$ is a Brownian motion (so is continuous), it follows that $Y_t$ is also a continuous function of time.

**8.9** Let $X_t$ be a standard Brownian motion. Compute the following conditional probability:

$$\mathbb{P}\{X_2 > 0 \mid X_1 > 0\}.$$

Are the events $\{X_1 > 0\}$ and $\{X_2 > 0\}$ independent?

**8.11** Let $X_t$ be a standard (one-dimensional) Brownian motion starting at 0 and let

$$M = \max\{X_t : 0 \le t \le 1\}.$$

Find the density for $M$ and compute its expectation and variance.

**8.14** Let $X_t$ be a standard (one-dimensional) Brownian motion started at a point $y$ chosen uniformly on the interval $(0,1)$. Suppose the motion is stopped whenever it reaches 0 or 1, and let $u(t,x), 0 < x < 1$ denote the density of the position $X_t$ restricted to those paths that have not left $(0,1)$. Find $u(t,x)$ explicitly in terms of an infinite series and use the series to find the function $h$ and the constant $\beta$ such that as $t \to \infty$,

$$u(t,x) \sim e^{-\beta t}\, h(x).$$

# Answers to Homework 10

Alireza Kazemipour

December 16, 2025

## Brownian Motion

### Question 9

Let $\mathbb{I}$ denote the indicator function. The event $\{X_1 > 0\}$ is a subset of the event $\{X_1\}$, hence we can construct the following relation using the tower rule of expectations:

$$\begin{aligned}\mathbb{P}(X_2 > 0 \mid X_1 > 0) &= \mathbb{E}\left[\mathbb{I}\left[X_2 > 0 \mid X_1 > 0\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left[X_2 > 0 \mid X_1\right]\right] \mid X_1 > 0\right] \\ &= \mathbb{E}\left[\mathbb{P}\left(X_2 > 0 \mid X_1\right) \mid X_1 > 0\right]\end{aligned}$$

To find $\mathbb{P}(X_2 > 0 \mid X_1)$, we write $X_2$ as

$$X_2 = (X_2 - X_1) + X_1.$$

Fix $X_1 = x \in \mathbb{R}$. Since $Z = (X_2 - X_1)$ is a zero mean normal random variable with the cumulative distribution function $\Phi$, we have

$$\begin{aligned}\mathbb{P}(X_2 > 0 \mid X_1 = x) &= \mathbb{P}(Z + x > 0) \\ &= \mathbb{P}(Z > -x) \\ &= \mathbb{P}(-Z < x) \\ &= \mathbb{P}(Z < x) && \text{(symmetry of the standard normal distribution)} \\ &= \Phi(x).\end{aligned}$$

Let $\phi(x) = \Phi'(x)$. Then, $\mathbb{E}\left[\mathbb{P}(X_2 > 0 \mid X_1) \mid X_1 > 0\right]$ is computed as:

$$\mathbb{E}\left[\mathbb{P}(X_2 > 0 \mid X_1) \mid X_1 > 0\right] = \frac{\mathbb{E}\left[\mathbb{P}(X_2 > 0 \mid X_1 > 0)\right]}{\mathbb{P}(X_1 > 0)} = \frac{\int_0^\infty \Phi(x)\phi(x)dx}{0.5}.$$

Now we solve the integral

$$\int_0^\infty \Phi(x)\phi(x)dx \overset{(u=\Phi(x))}{=} \int_{1/2}^1 udu = \frac{3}{8}.$$

Therefore, the final answer is equal to

$$\mathbb{E}\left[\mathbb{P}(X_2 > 0 \mid X_1) \mid X_1 > 0\right] = \frac{\int_0^\infty \Phi(x)\phi(x)dx}{0.5} = \frac{3/8}{0.5} = \frac{3}{4}.$$

Let $\mathcal{N}$ denote the normal distribution. To check the independence of $\{X_2 > 0\}$ and $\{X_1 > 0\}$, we have that $X_2 \sim \mathcal{N}(0,2)$ and $X_1 \sim \mathcal{N}(0,1)$, so $\mathbb{P}(X_2 > 0) = \mathbb{P}(X_1 > 0) = 0.5$. However,

$$\mathbb{P}(X_2 > 0 \mid X_1 > 0) = \frac{3}{8} \neq \mathbb{P}(X_2 > 0) = \frac{1}{2}.$$

Therefore, $\{X_2 > 0\}$ and $\{X_1 > 0\}$ are not independent.

## Question 11

Let $\Phi$ denote the standard normal cumulative distribution function (CDF). Using the reflection principle and the fact that $X_1 = X_1 - 0 = X_1 - X_0$ is a standard normal distribution, we have

$$\mathbb{P}\left(M \geq m\right) = 2\,\mathbb{P}\left(X_1 \geq m\right) = 2(1 - \Phi(m)).$$

Therefore,

$$\mathbb{P}\left(M \leq m\right) = 2\Phi(m) - 1.$$

So, the probability density function of $M$, $f_M$, is equal to

$$\frac{d}{dm}(2\Phi(m) - 1) = \sqrt{\frac{2}{\pi}}e^{-\frac{m^2}{2}}.$$

To find the mean of $M$, we have

$$\mathbb{E}\left[M\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty me^{-\frac{m^2}{2}}dm \overset{(mdm=dx)}{=} \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}dx = \sqrt{\frac{2}{\pi}}.$$

To find the variance of $M$, we compute the second moment of $M$:

$$\begin{aligned}
\mathbb{E}\left[M^2\right] &= \sqrt{\frac{2}{\pi}}\int_0^\infty m^2e^{-\frac{m^2}{2}}dm \\
&= \sqrt{\frac{2}{\pi}}\left(-me^{-\frac{m}{2}}\Big|_{m=0}^\infty + \int_0^\infty e^{-\frac{m^2}{2}}dm\right) && \text{(integration by parts)} \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{m^2}{2}}dm \\
&= \sqrt{\frac{2}{\pi}}\cdot\sqrt{\frac{\pi}{2}} && \text{(using the standard normal CDF)} \\
&= 1.
\end{aligned}$$

So, the variance is equal to

$$\mathsf{Var}(M) = \mathbb{E}\left[M^2\right] - \mathbb{E}\left[M\right]^2 = 1 - \frac{2}{\pi}.$$

## Question 14

This question is a slight modification of Example 2 of the textbook on page 187. So, we follow the pattern described for that example.

Let $g$ be the solution to the heat equation. We are looking for the steady-state equation $v$, where

$$v(x) = \lim_{t\to\infty} u(t,x), \quad \text{and} \quad u(t,x) = \mathbb{E}^y[g(X_t)].$$

Comparing our boundaries $\{0,1\}$ with the boundaries of Examples 2, $\{0,\pi\}$, we can conclude that $g$ is of the following form, where $n$ is replaced by $\pi n$ in the terms inside the summation:

$$g(x) = \sum_{n=1}^\infty C_ne^{-\frac{(\pi n)^2}{2}t}\sin(\pi nx).$$

Since

$$\int_0^1 \sin(\pi nx)\sin(\pi mx) = 0 \quad \text{if} \quad n \neq m,$$

We see that $C_n$ must satisfy

$$\int_0^1 g(x)\sin(\pi nx) = C_n\int_0^1 \sin^2(\pi nx) = C_n\int_0^1 \frac{1-\cos(2\pi nx)}{2} = \frac{C_n}{2}.$$

Since $g$ is the Dirac delta function at $y$, we have

$$C_n = 2\int_0^1 g(x)\sin(\pi n x) = 2\sin(\pi n y).$$

Hence, $g$ and $u$ are respectively of the following forms:

$$\begin{aligned}
g(x) &= 2\sum_{n=1}^{\infty} e^{-\frac{(\pi n)^2}{2}t}\sin(\pi n y)\sin(\pi n x), \\
u(t,x) &= \mathbb{E}^y[g(x)] \\
&= \int_0^1 2\sum_{n=1}^{\infty} e^{-\frac{(\pi n)^2}{2}t}\sin(\pi n y)\sin(\pi n x)dy \\
&= 2\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n\pi}\cdot e^{-\frac{(\pi n)^2}{2}t}\sin(\pi n x) \\
&= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k+1}\cdot e^{-\frac{(\pi(2k+1))^2}{2}t}\sin(\pi(2k+1)x).
\end{aligned}$$

So as $t\to\infty$, $k=0$ term dominates:

$$v(x) \sim \frac{4}{\pi}e^{-\frac{\pi^2}{2}t}\sin(\pi x).$$

Hence

$$\beta = \frac{\pi^2}{2} \quad \text{and} \quad h(x) = \frac{\pi}{4}\sin(\pi x).$$

**NAME**: ____________________ **TOTAL:** __________

**Final Exam for STAT 580, Stochastic Processes**



**Problem 1** Consider a Markov chain with transition matrix

$$\begin{bmatrix} 1-a & a & 0 \\ 0 & 1-b & b \\ c & 0 & 1-c \end{bmatrix}$$

where $0 < a, b, c < 1$. Find the stationary distribution.

**Problem 2** Let $p \in (0, 1)$. Consider a branching process with offspring distribution

$$\mathbb{P}(Y_0 = 0) = p^2, \quad \mathbb{P}(Y_0 = 1) = 2p(1-p), \quad \mathbb{P}(Y_0 = 2) = (1-p)^2.$$

Find the extinction probability.

**Problem 3** Failures occur for a mechanical process according to a Poisson process. Failures are classified as either major or minor. Major failures occur at the rate of 1.5 failures per hour. Minor failures occur at the rate of 3.0 failures per hour.

(a) Find the probability that two failures occur in 1 hour.

(b) Find the probability that in half an hour, no major failures occur.

(c) Find the probability that in 2 hours, at least two major failures occur or at least two minor failures occur.

**Problem 4** A facility has three machines and three mechanics. Machines break down at the rate of one per 24 hours. Breakdown times are exponentially distributed. The time it takes a mechanic to fix a machine is exponentially distributed with mean 6 hours. Only one mechanic can work on a failed machine at any given time. Let $X_t$ be the number of machines working at time $t$. Find the long-term probability that all machines are working.

**Problem 5** Consider the following Markov chain with state space $S = \{0, 1, ...\}$. A sequence of positive numbers $p_1, p_2, \cdots$ is given with $\sum_{i=1}^{\infty} p_i = 1$. Whenever the chain reaches state 0 it chooses a new state according to the $p_i$. Whenever the chain is at a state other than 0 it proceeds deterministically, one step at a time, toward 0. In other words, the chain has transition probability

$$p(x, x-1) = 1, x > 0, \quad p(0, x) = p_x, x > 0.$$

(1) Is this a recurrent chain since the chain keeps returning to 0.

(2) Under what conditions on the $p_x$ is the chain positive recurrent? In this case, what is the limiting probability distribution

**Problem 6** Let $(N(t), t \geq 0)$ be a Poisson process with parameter $\lambda$.

(a) Find a deterministic function $m(t)$ such that $M_t = (N_t - (\alpha t)^2 - m(t)$ is a martingale.

$(N_t - \lambda t)^2 - m(t)$

(b) For fixed integer $k > 0$, let $T = \min\{t : N_t = k\}$ be the first time $k$ arrivals occur for a Poisson process. Show that $T$ is a stopping time that satisfies the conditions of the optional stopping theorem.

Sampling

(c) Find the standard deviation of $T$.

**Problem 7** Let $X_1, X_2, \cdots$ be iid random variables with values in $S = \{-1, 0, 1, 2, \cdots\}$ with mean $\mu < 0$. Let $S_0 = 1$ and for $n > 0$

$$S_n = 1 + X_1 + \cdots + X_n .$$

Let $T = \min\{n; S_n = 0\}$.

(1) Explain why that $\mathbb{P}(T < \infty) = 1$

(2) Compute $\mathbb{E}(T)$.

**Problem 8** Let $a > 0$ and let $T$ be the first time that standard Brownian motion $(B_t, t \geq 0)$ exits the interval $(-a, a)$.

(a) Is it possible to find a deterministic function $\nu(t)$ such that

$$M(t) = B_t^4 - 6tB_t^2 + \nu(t)$$

is a martingale? If yes, find such $\nu(t)$.

(b) Show that $T < \infty$ and $T$ is a stopping time.

(c) Find the expected time $\mathbb{E}(T)$.

(d) Find the standard deviation of $T$.

# Answers To The Final Exam

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## Question 1

Let $\bar{\pi} = (\pi_1\ \pi_2\ \pi_3)$ denote the stationary distribution. From the equilibrium equation of finite Markov chains we have

$$\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\end{pmatrix} = \bar{\pi} = \bar{\pi}P = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3\end{pmatrix}\begin{pmatrix}1-a & a & 0\\ 0 & 1-b & b\\ c & 0 & 1-c\end{pmatrix}.$$

The above relation gives us the following set of equations:

$$\begin{cases}\pi_1 = (1-a)\pi_1 + c\pi_3\\ \pi_2 = (1-b)\pi_2 + a\pi_1\\ \pi_3 = (1-c)\pi_3 + b\pi_2\end{cases} = \begin{cases}a\pi_1 = c\pi_3\\ b\pi_2 = a\pi_1\\ c\pi_3 = b\pi_2\end{cases}.$$

We also know that since $\bar{\pi}$ is a probability distribution, it must be the case that

$$\pi_1 + \pi_2 + \pi_3 = 1.$$

Therefore, we have

$$\pi_1 + \frac{a}{b}\pi_1 + \frac{a}{c}\pi_1 = 1,$$

which results in

$$\pi_1 = \frac{bc}{bc + ac + ab}.$$

Then, the final answer is equal to

$$\pi_1 = \frac{bc}{bc+ac+ab}, \quad \pi_2 = \frac{ac}{bc+ac+ab}, \text{ and } \quad \pi_3 = \frac{ab}{bc+ac+ab}.$$

## Question 2

Let $a$ be the extinction probability. We know that

$$a = \phi(a) = \sum_{k=0}^{\infty} \mathbb{P}\left(Y_0 = k\right) a^k.$$

Therefore, we should solve

$$a = p^2 + 2p(1-p)a + (1-p)^2 a^2.$$

By solving the equation for $a$, we have

$$a = \frac{(1 - 2p + p^2) \pm (1-2p)}{2(1-p)^2} = \begin{cases}(\frac{p}{1-p})^2,\\ \text{or}\\ 1\end{cases}.$$

The extinction probability $a$ would the smaller of the obtained roots depending on the value of $p$, hence we have

$$a = \begin{cases}(\frac{p}{1-p})^2, & 0 \le p < \frac{1}{2};\\ 1, & \frac{1}{2} \le p \le 1.\end{cases}$$

## Question 3

Let $X_t$ and $Y_t$ represent the *independent* number of major and minor failures in interval $t$, where

$$\mathbb{P}(X_t = k) = e^{-1.5t} \cdot \frac{(1.5t)^k}{k!}, \quad \mathbb{P}(Y_t = k) = e^{-3t} \cdot \frac{(3t)^k}{k!}.$$

### Part a

$$\begin{aligned}\mathbb{P}(X_1 + Y_1 = 2) &= \mathbb{P}(X_1 = 0 \text{ and } Y_1 = 2) + \mathbb{P}(X_1 = 2 \text{ and } Y_1 = 0) + \mathbb{P}(X_1 = 1 \text{ and } Y_1 = 1) \\ &= \mathbb{P}(X_1 = 0)\,\mathbb{P}(Y_1 = 2) + \mathbb{P}(X_1 = 2)\,\mathbb{P}(Y_1 = 0) + \mathbb{P}(X_1 = 1)\,\mathbb{P}(Y_1 = 1) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(independence)} \\ &= 4.5e^{-4.5} + 1.125e^{-4.5} + 4.5e^{-4.5} \\ &= 10.125e^{-4.5}.\end{aligned}$$

### Part b

$$\mathbb{P}(X_{0.5} = 0) = e^{-0.75}.$$

### Part c

Let $A$ and $B$ be the events that respectively at least two major and minor failures in 2 hours occur. By De Morgan's laws we have

$$\mathbb{P}(A \cup B) = 1 - \mathbb{P}\left(A^{\complement} \cap B^{\complement}\right) \overset{\text{(independence)}}{=} 1 - \mathbb{P}\left(A^{\complement}\right)\mathbb{P}\left(B^{\complement}\right)$$

We have

$$\begin{aligned}\mathbb{P}\left(A^{\complement}\right) &= \mathbb{P}(X_2 = 0) + \mathbb{P}(X_2 = 1) = e^{-3} + 3e^{-3} = 4e^{-3}, \\ \mathbb{P}\left(B^{\complement}\right) &= \mathbb{P}(Y_2 = 0) + \mathbb{P}(Y_2 = 1) = e^{-6} + 6e^{-6} = 7e^{-6}.\end{aligned}$$

Therefore, the final answer is equal to

$$\mathbb{P}(A \cup B) = 1 - \mathbb{P}\left(A^{\complement}\right)\mathbb{P}\left(B^{\complement}\right) = 1 - 28e^{-9}.$$

## Question 4

We are dealing with a continuous-time finite state Markov chain. The state space is $\{0, 1, 2, 3\}$ representing the number of working machines. Let $\lambda_n$ and $\mu_n$ be the birth rate and death rate for state $n$. We know that the stationary distribution follows the relations:

$$q = 1 + \sum_{k=1}^{3} \Pi_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}, \quad \pi(n) = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} \cdot q^{-1}.$$

Let us compute $q$. The birth rate when there are $i$ working machines is equal to $(3 - i)\frac{1}{6}$. The death rate with there are $i$ working machines is equal to $\frac{i}{24}$. Therefore,

$$q = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = 1 + \frac{1/2}{1/24} + \frac{1/2 \times 1/3}{1/24 \times 2/24} + \frac{1/2 \times 1/3 \times 1/6}{1/24 \times 2/24 \times 3/24} = 125.$$

Hence, the stationary distribution that all machines are working is equal to

$$\pi(3) = \frac{1/2 \times 1/3 \times 1/6}{1/24 \times 2/24 \times 3/24} \cdot \frac{1}{125} = \frac{64}{125}.$$

### Question 5

### Part 1

Yes the chain is recurrent because there will always be a non-zero probability of returning to state zero and reaching any other state from state zero.

### Part 2

To have a positive recurrent chain, the expected return time, $\mathbb{E}[T_x]$, to any state $x \in \{0, 1, \dots\}$ should be finite. Since the chain is recurrent, it suffices to focus on only state zero because all other states can be reached from this state. For state zero we should have

$$\mathbb{E}[T_0] = 1 + \sum_{x=1}^{\infty} x \cdot p_x.$$

Thus, to have $\mathbb{E}[T_0] < \infty$, we should have $\sum_{x=1}^{\infty} x \cdot p_x < \infty$. Hence, the condition is $p_x \in o\left(\frac{1}{x^2}\right)$.

Now, we compute the stationary distribution $\pi$. For state zero, we have

$$\pi(0) = \pi(1).$$

For states $1, 2$, and $3$ we have:

$$\begin{aligned}\pi(1) &= \pi(0)p_1 + \pi(2)\\ \pi(2) &= \pi(0)p_2 + \pi(3)\\ \pi(3) &= \pi(0)p_3 + \pi(4).\end{aligned}$$

So, we conclude that

$$\pi(x) = \pi(0)\sum_{j=x}^{\infty} p_j, \quad x \geq 1.$$

We know that since $\pi$ is a probability distribution, it must be the case that

$$\pi(0) + \sum_{x=1}^{\infty} \pi(x) = 1.$$

Therefore, we have

$$\pi(0) + \pi(0)\sum_{x=1}^{\infty}\sum_{i=x}^{\infty} p_i = 1,$$

which is equal to solving

$$\begin{aligned}\pi(0) + \pi(0)\sum_{i=1}^{\infty}\sum_{x=1}^{i} p_i &= 1\\ \pi(0)\left(1 + \sum_{i=1}^{\infty} i \cdot p_i\right) &= 1\\ \pi(0) &= \frac{1}{1 + \sum_{i=1}^{\infty} i \cdot p_i}.\end{aligned}$$

We computed $\pi(0)$, now for states $x \geq 1$ we have

$$\pi(x) = \frac{\sum_{j=x}^{\infty} p_j}{1 + \sum_{i=1}^{\infty} i \cdot p_i}.$$

## Question 6

### Part a

Let $s \leq t$. We have to show

$$\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = M_s.$$

Define

$$A := N_s - \lambda s, \quad B := N_t - N_s - \lambda(t-s).$$

We have that

$$\begin{aligned}
\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[(A+B)^2 \mid \mathcal{F}_s\right] - m(t) \\
&= \mathbb{E}\left[A^2 \mid \mathcal{F}_s\right] + 2\,\mathbb{E}\left[AB \mid \mathcal{F}_s\right] + \mathbb{E}\left[B^2 \mid \mathcal{F}_s\right] - m(t) \\
&= A^2 + 2A\,\mathbb{E}\left[B \mid \mathcal{F}_s\right] + \mathbb{E}\left[B^2 \mid \mathcal{F}_s\right] - m(t).
\end{aligned}$$

Hence, we need to compute the first and the second moment of $B$ given $\mathcal{F}_s$. First, let us determine the distribution of $B$ given $\mathcal{F}_s$. Let $\tau := t - s$ and $Z_\tau = N_{\tau+s} - N_s$. The independence of increments in different intervals in a Poisson process implies that $Z_\tau$ is a Poisson with rate $\lambda$, where $\mathbb{E}\left[Z_\tau \mid \mathcal{F}_s\right] = \lambda\tau$. Hence, we have

$$\mathbb{E}\left[B \mid \mathcal{F}_s\right] = \mathbb{E}\left[Z_\tau \mid \mathcal{F}_s\right] - \lambda\tau = 0.$$

To compute the second moment of $B$ given $\mathcal{F}_s$, we have

$$\mathbb{E}\left[B^2 \mid \mathcal{F}_s\right] = \mathbb{E}\left[Z_\tau^2 \mid \mathcal{F}_s\right] - (\lambda\tau)^2.$$

Since $Z_\tau^2 = Z_\tau(Z_\tau - 1) + Z_\tau$, by using the law of the unconscious statistician, we have

$$\begin{aligned}
\mathbb{E}\left[Z_\tau^2 \mid \mathcal{F}_s\right] &= \mathbb{E}\left[Z_\tau(Z_\tau - 1) \mid \mathcal{F}_s\right] + \mathbb{E}\left[Z_\tau \mid \mathcal{F}_s\right] \\
&= \sum_{k=0}^{\infty} k(k-1) e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} + \lambda\tau \\
&= (\lambda\tau)^2 e^{-\lambda\tau} \sum_{k=2}^{\infty} \frac{(\lambda\tau)^{k-2}}{(k-2)!} + \lambda\tau \\
&= (\lambda\tau)^2 e^{-\lambda\tau} e^{\lambda\tau} + \lambda\tau \\
&= (\lambda\tau)^2 + \lambda\tau.
\end{aligned}$$

Hence,

$$\mathbb{E}\left[B^2 \mid \mathcal{F}_s\right] = \mathbb{E}\left[Z_\tau^2 \mid \mathcal{F}_s\right] - (\lambda\tau)^2 = \lambda\tau.$$

Therefore,

$$\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = A^2 + \lambda\tau = A^2 + \lambda(t-s).$$

To make $M_t$ a martingale we must have

$$\mathbb{E}\left[M_t \mid \mathcal{F}_s\right] = A^2 + \lambda(t-s) - m(t) = M_s = A^2 - m(s).$$

The above relation gives us $m(t) - m(s) = \lambda(t-s)$, or in other words

$$m'(t) = \lambda.$$

Hence,

$$m(t) = \lambda t + c, \quad c \in \mathbb{R}.$$

## Part b

We need to verify the followings:

1. At every time step $t$, $\{T = t\}$ is $\mathcal{F}_t$-measurable.
2. $\mathbb{P}(T < \infty) = 1$.

At every timestep $t$, since the value of $N_t$ is completely known with the information available at time $t$, it is known whether $k$ arrivals has occurred or not without looking into the future. Hence, $\{T = t\}$ is $\mathcal{F}_t$-measurable.

Let $T_i$ denote the $i$th arrival time that has the exponential distribution with parameter $\lambda$. We can write $T$ as

$$T = T_1 + T_2 + \cdots + T_k.$$

On the other hand,

$$\mathbb{P}(T < \infty) = 1 - \mathbb{P}(T = \infty) = 1 - \mathbb{P}\left(\cup_{i=1}^{k} \{T_i = \infty\}\right) \overset{\text{(independence)}}{=} 1 - \sum_{i=1}^{k} \mathbb{P}(T_i = \infty).$$

But since $k$ is finite, $\lambda > 0$, and $T_i$ has the exponential distribution for each $i$, $\mathbb{P}(T_i = \infty)$ is almost surely zero. Hence,

$$\mathbb{P}(T < \infty) = 1 - \mathbb{P}(T = \infty) = 1.$$

## Part c

We want to use the following relation for the variance and take its square root in the end to get the standard deviation:

$$\mathbb{V}[T] = \mathbb{E}\left[T^2\right] - \mathbb{E}[T]^2.$$

Consider the martingale $M_t = N_t - \lambda t$. Suppose $M_0, M_1, \ldots$ are uniformly integrable (to be proven later). Since $T$ satisfies the optional sampling theorem conditions and $\mathbb{E}\left[|M_T| = |k - \lambda T|\right] < \infty$, we can use the optional sampling theorem that

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = k - \lambda \mathbb{E}[T].$$

Hence,

$$\mathbb{E}[T] = \frac{k}{\lambda}.$$

Consider the martingale $M_t = (N_t - \lambda t)^2 - \lambda t$. In Part (a) we proved that $M_t$ is a martingale. Suppose $M_0, M_1, \ldots$ are uniformly integrable (to be proven later). Since $T$ satisfies the optional sampling theorem conditions and $\mathbb{E}\left[|M_T| = |(k - \lambda T)^2 - \lambda T|\right] < \infty$, we can use the optional sampling theorem that

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}\left[(k - \lambda T)^2 - \lambda T\right].$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[(k - \lambda T)^2 - \lambda T\right] &= 0 \\
\mathbb{E}\left[k^2 - \lambda T(2k + 1) + \lambda^2 T^2\right] &= 0 \\
k^2 - k(2k + 1) + \lambda^2 \mathbb{E}\left[T^2\right] &= 0 \\
-k^2 - k + \lambda^2 \mathbb{E}\left[T^2\right] &= 0 \\
\mathbb{E}\left[T^2\right] &= \frac{k^2 + k}{\lambda^2}.
\end{aligned}$$

Therefore,

$$\mathbb{V}[T] = \frac{k^2 + k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}.$$

the standard deviation is equal to:

$$\frac{\sqrt{k}}{\lambda}.$$

Now we need to show that $X_t = N_t - \lambda t$ and $Y_t = (N_t - \lambda t)^2 - \lambda t$ are uniformly integrable. It suffices to show that there exits constants $C_1, C_2$ such that

$$\mathbb{E}\left[X_t^2\right] < C_1, \quad \mathbb{E}\left[Y_t^2\right] < C_2.$$

For $X_t$ we have:

$$\begin{aligned}
\mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[(N_t - \lambda t)^2\right] \\
&\leq \mathbb{E}\left[(N_T + \lambda T)^2\right] && \text{(we stop at } T\text{)} \\
&= \mathbb{E}\left[(k + \lambda T)^2\right] \\
&= k^2 + 2\lambda \mathbb{E}\left[T\right] + \lambda^2 \mathbb{E}\left[T^2\right] \\
&< \infty. && (T \text{ is almost surely bounded})
\end{aligned}$$

For $Y_t$ we have:

$$\begin{aligned}
\mathbb{E}\left[Y_t^2\right] &= \mathbb{E}\left[((N_t - \lambda t)^2 - \lambda t)^2\right] \\
&\leq \mathbb{E}\left[((N_T + \lambda T)^2 + \lambda T)^2\right] && \text{(we stop at } T\text{)} \\
&= \mathbb{E}\left[(k + \lambda T)^4 + 2(k + \lambda T)\lambda T + \lambda^2 T^2\right] \\
&< \infty. && (T \text{ is almost surely bounded})
\end{aligned}$$

For the sake of brevity and avoiding the repetition of steps that we went through for $X_t$, we omitted the calculations that explicitly lead to the terms that are moments of $T$ in the proof of $Y_t$'s uniform integrability. Nonetheless, since $T$ is almost surely bounded, all of its moments are bounded, which completes the its corresponding proof.

# Question 7

## Part 1

By the strong law of large numbers:

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1 + X_1 + X_2 + \cdots + X_n}{n} \approx \mu.$$

Hence for large values of $n$,

$$S_n = \mu n = -\infty.$$

Since $S_0 = 1$ and $S_\infty = -\infty$, there must be a time $n_0$, where at $n_0$, $S_{n_0}$ crosses 0. Now we prove that not only $S_n$ crosses 0 but it also hits it with probability one. We can write $S_{n_0}$ as

$$S_{n_0} = S_{n_0-1} + X_{n_0}.$$

Consider a crossover scenario; suppose $S_{n_0} \leq 0$ so $S_{n_0-1} \geq 1$. Since $X_{n_0} \geq -1$, we have

$$S_{n_0} = S_{n_0-1} + X_{n_0} \geq 1 - 1 \geq 0.$$

Because we had assumed that $S_{n_0} \leq 0$ and we obtained $S_{n_0} \geq 0$, it must be the case that $S_{n_0} = 0$. Hence, with probability one, there exists a time $T = n_0 < \infty$, where $S_T = 0$.

## Part 2

$$\begin{aligned}
\mathbb{E}\left[S_n\right] &= 1 + \mathbb{E}\left[X_1 + X_2 + \cdots + X_n\right] \\
\mathbb{E}\left[\mathbb{E}\left[S_n \mid n = T\right]\right] &= 1 + \mathbb{E}\left[\mathbb{E}\left[X_1 + X_2 + \cdots + X_n \mid n = T\right]\right] && \text{(Tower rule)} \\
\mathbb{E}\left[S_T\right] &= 1 + \mathbb{E}\left[\mu T\right] \\
0 &= 1 + \mathbb{E}\left[\mu T\right] \\
-\frac{1}{\mu} &= \mathbb{E}\left[T\right].
\end{aligned}$$

## Question 8

### Part a

Let $s \le t$ and $Y = B_t - B_s$. $Y$ is a normal distribution with mean 0, variance $t-s$, and moments

$$\begin{aligned}
\mathbb{E}[Y] &= 0 \\
\mathbb{E}\left[Y^2\right] &= t-s \\
\mathbb{E}\left[Y^3\right] &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} y^3 e^{-\frac{y^2}{2(t-s)}}\, dy = 0 \\
\mathbb{E}\left[Y^4\right] &= \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} y^4 e^{-\frac{y^2}{2(t-s)}}\, dy = 3(t-s)^2.
\end{aligned}$$

We have:

$$\begin{aligned}
M_s &= B_s^4 - 6sB_s^2 + \nu(s), \\
M_t &= (B_s + Y)^4 - 6t(B_s+Y)^2 + \nu(t) \\
&= B_s^4 + 4B_s^3 Y + 6B_s^2 Y^2 + 4B_s Y^3 + Y^4 - 6tB_s^2 - 12tB_sY - 6tY^2 + \nu(t).
\end{aligned}$$

Now we want to impose the martingale property:

$$\begin{aligned}
\mathbb{E}[M_t \mid \mathcal{F}_s] = B_s^4 &+ 4B_s^3\,\mathbb{E}[Y \mid \mathcal{F}_s] + 6B_s^2\,\mathbb{E}\left[Y^2 \mid \mathcal{F}_s\right] + \\
&4B_s\,\mathbb{E}\left[Y^3 \mid \mathcal{F}_s\right] + \mathbb{E}\left[Y^4 \mid \mathcal{F}_s\right] - 6tB_s^2 - 12tB_s\,\mathbb{E}[Y \mid \mathcal{F}_s] - 6t\,\mathbb{E}\left[Y^2 \mid \mathcal{F}_s\right] + \nu(t) \\
&= B_s^4 - 6sB_s^2 + 3s^2 - 3t^2 + \nu(t).
\end{aligned}$$

Therefore, we solve $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$:

$$\begin{aligned}
B_s^4 - 6sB_s^2 + 3s^2 - 3t^2 + \nu(t) &= B_s^4 - 6sB_s^2 + \nu(s) \\
\nu(t) - \nu(s) &= 3t^2 - 3s^2.
\end{aligned}$$

Hence it is enough to choose $\nu(t) = 3t^2 + c,\ c \in \mathbb{R}$.

### Part b

Since paths are continuous in time for Brownian motions, the event $\{|B_t| > a\}$ is measurable using the location of the Brownian motion at time $t$, hence $\{|B_t| > a\}$ is $\mathcal{F}_t$-measurable. Consequently, for any $T = t$, the event $\{|B_T| > a\}$ is $\mathcal{F}_t$-measurable, therefore $T$ is a stopping time.

Now we show that $T < \infty$ almost surely. For simplicity we will only show that the Brownian motion eventually hits the upper limit $a$. The argument for hitting the lower limit, $-a$, follows similarly.

Fix a positive real number $t$ such that $B_t \ge a$. We have:

$$\begin{aligned}
\mathbb{P}(B_t \ge a) &= \mathbb{P}(B_t \ge a \mid T \le t) \cdot \mathbb{P}(T \le t) \\
&= \mathbb{P}(B_t \ge B_T \mid T \le t) \cdot \mathbb{P}(T \le t) \\
&= \mathbb{P}(B_t - B_T \ge 0 \mid T \le t) \cdot \mathbb{P}(T \le t) \\
&= \frac{1}{2} \cdot \mathbb{P}(T \le t),
\end{aligned}$$

where we used the fact that $B_t - B_T$ is a normal random variable with mean zero (so is symmetric around the origin). Therefore,

$$\mathbb{P}(T \le t) = \mathbb{P}(T < t) = 2\,\mathbb{P}(B_t \ge a) = 2\int_a^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx = 2\int_{\frac{a}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du.$$

Hence,

$$\mathbb{P}(T < \infty) = \lim_{t\to\infty} 2\int_{\frac{a}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du = 2 \times \frac{1}{2} = 1.$$

## Part c

Consider the martingale $M_t = B_t^2 - t$. Each random variable $M_0, M_1, \ldots$ is uniformly integrable (the proof is similar to what we did in Part (c) of Question 6). Since $T$ satisfies the optional sampling theorem conditions and $\mathbb{E}\left[|M_T| = |a^2 - T|\right] < \infty$, we can use the optional sampling theorem:

$$0 = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_T\right] = a^2 - \mathbb{E}\left[T\right].$$

Hence,

$$\mathbb{E}\left[T\right] = a^2.$$

## Part d

Consider the martingale $M_t = B_t^4 - 6tB_t^2 + 3t^2$. In Part (a) we proved that $M_t$ is a martingale. $M_0, M_1, \ldots$ are uniformly integrable (the proof is similar to what we did in Part (c) of Question 6). Since $T$ satisfies the optional sampling theorem conditions and $\mathbb{E}\left[|M_T| = |a^4 - 6Ta^2 + 3T^2|\right] < \infty$, we can use the optional sampling theorem that

$$0 = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_T\right] = \mathbb{E}\left[a^4 - 6Ta^2 + 3T^2\right].$$

Hence,

$$\begin{aligned} \mathbb{E}\left[a^4 - 6Ta^2 + 3T^2\right] &= 0 \\ a^4 - 6a^2\,\mathbb{E}\left[T\right] + 3\,\mathbb{E}\left[T^2\right] &= 0 \\ -5a^4 + 3\,\mathbb{E}\left[T^2\right] &= 0 \\ \mathbb{E}\left[T^2\right] &= \frac{5a^4}{3}. \end{aligned}$$

Therefore,

$$\mathbb{V}[T] = \frac{5a^4}{3} - a^4 = \frac{2a^4}{3}.$$

The standard deviation is equal to

$$\sqrt{\mathbb{V}[T]} = a^2\sqrt{\frac{2}{3}}.$$